

SINGULARITY CATEGORIES OF REPRESENTATIONS OF QUIVERS OVER LOCAL RINGS

MING LU

ABSTRACT. Let Λ° be a finite-dimensional algebra with finite global dimension, $R_k = K[X]/(X^k)$ be the \mathbb{Z} -graded local ring, and $H^k = \Lambda^\circ \otimes_K R_k$. We consider the singularity category $D_{sg}(\text{mod}^{\mathbb{Z}}(H^k))$ of the graded modules over H^k . It is obtained that there is a tilting object in $D_{sg}(\text{mod}^{\mathbb{Z}}(H^k))$ such that its endomorphism algebra is isomorphic to the triangular matrix algebra $T_{k-1}(\Lambda^\circ)$ with coefficients in Λ° and there is a triangle equivalence between $D_{sg}(\text{mod}^{\mathbb{Z}/k\mathbb{Z}}(H^k))$ and the root category of $T_{k-1}(\Lambda^\circ)$. Furthermore, a classification of H^k up to Cohen-Macaulay representation type is given for Λ° to be hereditary.

1. INTRODUCTION

The singularity category of an algebra is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to bounded complexes of finitely generated projective modules [7], see also [24]. Recently, D. Orlov's global version [41] attracted a lot of interest in algebraic geometry and theoretical physics. In particular, the singularity category measures the homological singularity of an algebra [24]: the algebra has finite global dimension if and only if its singularity category is trivial.

Gorenstein (also called Iwanaga-Gorenstein) algebra A , where by definition A has finite injective dimension both as a left and a right A -module, is inspired from commutative ring theory. A fundamental result of R. Buchweitz [7] and D. Happel [24] states that for a Gorenstein algebra A , the singularity category is triangle equivalent to the stable category of (maximal) Cohen-Macaulay (also called Gorenstein projective) A -modules, which generalized J. Rickard's result [44] on self-injective algebras.

Inspired by the classification of the modules over finite-dimensional algebras over an algebraically closed field [11, 10], Y. A. Drozd and G. M. Greuel introduced the Cohen-Macaulay module type for Cohen-Macaulay algebras or orders [12, 13], which is also suitable for the classification of Cohen-Macaulay modules over any algebras. It is claimed that with respect to the classification of Cohen-Macaulay modules over any algebra A split into three types:

- Cohen-Macaulay finite, when A has only a finite number of indecomposable Cohen-Macaulay modules;
- Cohen-Macaulay tame, when the indecomposable modules of any fixed dimension form a finite number of 1-parameter families, together with, maybe, a finite set of "isolated" modules;
- Cohen-Macaulay wild, which can be defined in two ways: either as those algebras having n -parameter families of indecomposable modules of a fixed dimension for arbitrary n (this is called Cohen-Macaulay semi-wild in the following) or as those for which the classification of modules includes the classification of representations of all algebras (this is just called Cohen-Macaulay wild).

2000 *Mathematics Subject Classification.* 16E45, 18E30, 18E35.

Key words and phrases. Representations of quivers over local rings; Gorenstein algebras; Orbit categories; Singularity categories.

This classification is called the Tame-wild dichotomy for Cohen-Macaulay modules, and are verified in many cases, and no counterexamples has been found.

Another motivation of this paper is the root category of a finite-dimensional algebra. Root category was first introduced by Happel [22] for finite-dimensional hereditary algebra. Let A be a finite-dimensional hereditary algebra over a field K . Let $D^b(\text{mod } A)$ be the derived category of finitely generated right A -modules. Then the root category \mathcal{R}_A of A is defined to be the 2-periodic orbit category $D^b(\text{mod } A)/\Sigma^2$, where Σ is the suspension functor. It was proved by L. Peng and J. Xiao [42] that the root category \mathcal{R}_A is a triangulated category. With this triangle structure, Peng and Xiao [43] constructed a so called Ringel-Hall Lie algebra associated to each root category and realized all the symmetrizable Kac-Moody Lie algebras. In general, for A not hereditary, $D^b(\text{mod } A)/\Sigma^2$ is not triangulated, with the help of the DG orbit categories defined in [29], one can construct another 2-periodic triangulated category which is a triangulated hull of $D^b(\text{mod } A)/\Sigma^2$. This triangulated hull is also called the root category of A and denoted also by \mathcal{R}_A , see e.g. [15].

Let Λ° be a finite-dimensional algebra with finite global dimension, $R_k = K[X]/(X^k)$ be the \mathbb{Z} -graded local ring with X degree one for a positive integer k , and $H^k = \Lambda^\circ \otimes_K R_k$. Then H^k is a positively graded Gorenstein algebra. C. M. Ringel and M. Schmidmeier [47] investigate the Cohen-Macaulay modules over H^k with Λ° hereditary of type A_2 by using submodule categories, and describe its Auslander-Reiten quiver for $k < 6$. Later, C. M. Ringel and P. Zhang [48] prove that the singularity category of H^2 with $\Lambda^\circ = KQ^\circ$ hereditary is triangle equivalent to the triangulated orbit category $D^b(\text{mod } \Lambda^\circ)/\Sigma$. X.-H. Luo and P. Zhang generalize these works and introduce monomorphism categories over finite acyclic quivers to describe the Cohen-Macaulay modules over $A \otimes_K KQ$ (also $A \otimes_K KQ/I$ with I generated by monomial relations) with A a coherent algebra [37, 38]. Recently, D. Shen describes Gorenstein projective modules over the tensor product of two algebras in terms of their underlying onesided modules [49].

In this paper, we mainly consider the singularity category $D_{sg}(\text{mod}^{\mathbb{Z}}(H^k))$ of the graded modules over H^k .

Yamaura [52] proved that for a positively graded selfinjective algebra, its stable category of the \mathbb{Z} -graded modules admits a tilting module. Following him, we prove that the singularity category $D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$ of the \mathbb{Z} -graded modules over H^k has a tilting module with the same construction, in particular, its endomorphism algebra is isomorphic to the triangular matrix algebra $T_{k-1}(\Lambda^\circ)$ with coefficients in Λ° . In this way, we get that $D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$ is triangle equivalent to the bounded derived category $D^b(T_{k-1}(\Lambda^\circ))$. Viewing H^k as a $\mathbb{Z}/k\mathbb{Z}$ -graded algebra naturally and considering the singularity category $D_{sg}(\text{mod}^{\mathbb{Z}/k\mathbb{Z}} H^k)$, we prove that the triangle equivalence above induces a triangle equivalence between $D_{sg}(\text{mod}^{\mathbb{Z}/k\mathbb{Z}} H^k)$ and the root category of $T_{k-1}(\Lambda^\circ)$. Furthermore, when $k = 2$, this result reappears the result of [48], which states: $D_{sg}(\text{mod } H^2)$ is triangle equivalent to the triangulated orbit category $D^b(\text{mod } \Lambda^\circ)/\Sigma$ if Λ° is hereditary. Finally, we classify H^k for Λ° hereditary up to the Cohen-Macaulay module type, and give some examples to describe the Auslander-Reiten quivers for some H^k of Cohen-Macaulay finite type.

Acknowledgments. The author deeply thanks Professor Bin Zhu for his guidance, inspiration, helpful discussion and comments. The work was done during the stay of the first author at the Department of Mathematics, University of Bielefeld. He is deeply indebted to Professor Henning Krause for his kind hospitality, inspiration and continuous encouragement. The author also deeply thanks Professor H. Lenzing for helpful discussion on the tubular algebras. The author was supported by the National Natural Science Foundation of China (No. 11401401 and No. 11601441).

2. PRELIMINARIES

Throughout this paper K is an algebraically closed field unless specified. We denote by D the K -dual, i.e. $D(-) = \text{Hom}_K(-, K)$.

Let A be a K -algebra. We denote by $\text{mod } A$ the category of finitely generated (right) modules, by $\text{proj } A$ the category of finitely generated projective A -modules.

For an additive category \mathcal{A} , we use $\text{Ind } \mathcal{A}$ to denote the set of all non-isomorphic indecomposable objects in \mathcal{A} .

2.1. Group graded algebras. Let G be an abelian group, and $A = \bigoplus_{i \in G} A_i$ be a G -graded algebra. A G -graded A -module X is of form $\bigoplus_{i \in G} X_i$, where X_i is the degree i part of X . The category $\text{mod}^G A$ of (finitely generated) G -graded A -modules is defined as follows.

- The objects are G -graded A -modules,
- For G -graded A -modules X and Y , the morphism space from X to Y in $\text{mod}^G A$ is defined by

$$\text{Hom}_{\text{mod}^G A}(X, Y) := \text{Hom}_A(X, Y)_0 := \{f \in \text{Hom}_A(X, Y) \mid f(X_i) \subseteq Y_i \text{ for any } i \in G\}.$$

We denote by $\text{proj}^G A$ the full subcategory of $\text{mod}^G A$ consisting of projective objects.

For $i \in G$, we denote by $(i) : \text{mod}^G A \rightarrow \text{mod}^G A$ the *grade shift functor*. Then for any two G -graded A -modules X, Y ,

$$\text{Hom}_A(X, Y) = \bigoplus_{i \in G} \text{Hom}_A(X, Y(i))_0.$$

Denote by J_A the *Jacobson radical* of A .

Proposition 2.1 ([52]). *Assume that $J_A = J_{A_0} \oplus (\bigoplus_{i \in G \setminus \{0\}} A_i)$. We take a set \overline{PI} of idempotents of A_0 such that $\{eA_0 \mid e \in \overline{PI}\}$ is a complete list of indecomposable projective A_0 -modules. Then the following assertions hold.*

- Any complete set of orthogonal primitive idempotents of A_0 is that of A .
- A complete list of simple objects in $\text{mod}^G A$ is given by

$$\{S(i) \mid i \in G, S \text{ is a simple } A_0\text{-module}\}.$$

- A complete list of indecomposable projective objects in $\text{mod}^G A$ is given by

$$\{eA(i) \mid i \in G, e \in \overline{PI}\}.$$

- A complete list of indecomposable injective objects in $\text{mod}^G A$ is given by

$$\{D(Ae)(i) \mid i \in G, e \in \overline{PI}\}.$$

Let A be a positively graded algebra, i.e., $A = \bigoplus_{i \geq 0} A_i$, and a a positive integer. Then we can regard A as a $\mathbb{Z}/a\mathbb{Z}$ -graded algebra by

$$A_{\bar{i}} = \bigoplus_{j \in i + a\mathbb{Z}} A_j$$

for any $i \in \mathbb{Z}$. Also we regard $X \in \text{mod}^{\mathbb{Z}} A$ as a $\mathbb{Z}/a\mathbb{Z}$ -graded A -module by

$$X_{\bar{i}} = \bigoplus_{j \in i + a\mathbb{Z}} X_j.$$

In this way there is an additive exact functor

$$F_a : \text{mod}^{\mathbb{Z}} A \rightarrow \text{mod}^{\mathbb{Z}/a\mathbb{Z}} A.$$

Furthermore, let X and Y be \mathbb{Z} -graded A -modules. Then

$$\mathrm{Hom}_A(F_a(X), F_a(Y))_{\bar{0}} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(X, Y(ia))_0.$$

If $a = 1$, then $\mathrm{mod}^{\mathbb{Z}/a\mathbb{Z}} A$ coincides with $\mathrm{mod} A$, and F_a is just the forgetful functor from $\mathrm{mod}^{\mathbb{Z}} A$ to $\mathrm{mod} A$.

Note that for a positively graded algebra A , the equation

$$J_A = J_{A_0} \oplus \left(\bigoplus_{i \neq 0} A_i \right) = J_{A_{\bar{0}}} \oplus \left(\bigoplus_{\bar{i} \neq 0} A_{\bar{i}} \right)$$

always holds. So A with both gradings above satisfies the assumption of Proposition 2.1, see [52, Proposition 2.18].

2.2. Gorenstein algebra and Singularity category.

Definition 2.2. A K -algebra A is called a *Gorenstein (or Iwanaga-Gorenstein) algebra* if A satisfies $\mathrm{inj. dim} {}_A A < \infty$ and $\mathrm{inj. dim} A_A < \infty$.

Then a K -algebra A is Gorenstein if and only if $\mathrm{inj. dim} A_A < \infty$ and $\mathrm{proj. dim} D({}_A A) < \infty$. Observe that for a Gorenstein algebra A , we have $\mathrm{inj. dim} {}_A A = \mathrm{inj. dim} A_A$, see [22, Lemma 6.9], the common value is denoted by $\mathrm{G. dim} A$. If $\mathrm{G. dim} A \leq d$, we say that A is a *d-Gorenstein algebra*.

Definition 2.3. Let A be a Gorenstein algebra. A finitely generated A -module M is called *(maximal) Cohen-Macaulay (or Gorenstein projective)* if

$$\mathrm{Ext}_A^i(M, A) = 0 \text{ for } i \neq 0.$$

The full subcategory of Cohen-Macaulay modules in $\mathrm{mod} A$ is denoted by $\mathrm{CM}(A)$.

Theorem 2.4 ([7, 2, 14]). Let A be a d -Gorenstein algebra. Then $\mathrm{CM}(A) = \Omega^d(\mathrm{mod} A)$, where Ω is the syzygy functor. Furthermore, a module M is Cohen-Macaulay if and only if there is an exact sequence $0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$ with each P^i projective.

Similarly, for G -graded algebras where G is an abelian group, one defines the notion of *G-graded Gorenstein algebras*. Thanks to a remark by Van den Bergh [51, p.670, line 9-11], we infer that a \mathbb{Z} -graded A -module M has finite injective dimension in $\mathrm{mod}^{\mathbb{Z}} A$ if and only if M has finite injective dimension as an ungraded module. Thus the \mathbb{Z} -graded algebra A is graded Gorenstein if and only if it is Gorenstein as an ungraded ring.

Let A be a G -graded Gorenstein algebra. A G -graded A -module M is called *G-graded (maximal) Cohen-Macaulay* if M satisfies that

$$\mathrm{Ext}_{\mathrm{mod}^G A}^i(M, A(j)) = 0 \text{ for } i \neq 0, j \in G.$$

It is worth noting that the graded version of Theorem 2.4 is valid. We denote by $\mathrm{CM}^G(A)$ the full subcategory of $\mathrm{mod}^G A$ formed by all G -graded Cohen-Macaulay modules. Then $\mathrm{CM}^G(A)$ is a Frobenius category, and $\underline{\mathrm{CM}}^G(A)$ has a structure of a triangulated category whose shift functor is given by the graded cosyzygy functor Ω^{-1} [7, 22].

Analogous to the representation type of algebras [10, 11, 12, 13], an algebra A is of *finite Cohen-Macaulay type*, or simply, *CM-finite*, if there are only finitely many isomorphism classes of indecomposable finitely generated Cohen-Macaulay modules. A is of *tame Cohen-Macaulay type*, or simply, *CM-tame* if A is not CM-finite, whereas for any dimension $d > 0$, there are finite number of $K[T]$ - A -bimodules M_i which are finitely generated free as left $K[T]$ -modules such that all but a finite many indecomposable Cohen-Macaulay A -modules of dimension d are

isomorphic to $K[T]/(T - \lambda) \otimes_{K[T]} M_i$ for $\lambda \in K$. We say that A is of *wild Cohen-Macaulay type* or A is *CM-wild* if there is a finitely generated $K\langle X, Y \rangle$ - A -bimodule B which is free as a left $K\langle X, Y \rangle$ -module such that the functor $- \otimes_{K\langle X, Y \rangle} B$ from $\text{mod } K\langle X, Y \rangle$, the category of finitely generated $K\langle X, Y \rangle$ -modules, to $\text{CM } A$ preserves indecomposability and reflects isomorphisms.

We say that A is of *semi-wild Cohen-Macaulay type* or A is *CM-semi-wild* if there is a finitely generated $K\langle X, Y \rangle$ - A -bimodule B which is free as a left $K\langle X, Y \rangle$ -module such that the functor $- \otimes_{K\langle X, Y \rangle} B$ from $\text{mod } K\langle X, Y \rangle$ to $\text{CM } A$ satisfies that for any $K\langle X, Y \rangle$ -module M there exists only a finite number (up to isomorphism) of such $K\langle X, Y \rangle$ -modules N that $M \otimes_{K\langle X, Y \rangle} B \cong N \otimes_{K\langle X, Y \rangle} B$.

By [18, 11, 12, 13], A is CM-wild if for every finitely generated K -algebra Λ there is an exact functor from the category of finite-dimensional representation of Λ to the category of Cohen-Macaulay A -modules, which maps non-isomorphic modules to non-isomorphic ones and indecomposable to indecomposable. A is CM-semi-wild if for every finitely generated K -algebra Λ there is an exact functor F from the category of finite-dimensional representation of Λ to the category of Cohen-Macaulay A -modules, which maps indecomposable to indecomposable and for any right Λ -module M , there are only finitely many $N \in \text{mod } \Lambda$ such that $F(M) \cong F(N)$. A is CM-tame if and only if the indecomposable modules of any fixed dimension d form a finite number of 1-parameter families, together with, maybe, a finite set of “isolated” modules.

Lemma 2.5 ([13]). *For any finite-dimensional algebra A , A can not be both of Cohen-Macaulay wild type and of Cohen-Macaulay tame type.*

Definition 2.6 ([41]). *Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded algebra. The singularity category is defined to be the Verdier localization*

$$D_{sg}(\text{mod } A) := D^b(\text{mod } A) / D^b(\text{proj } A),$$

the G -graded singularity category is

$$D_{sg}(\text{mod}^G A) := D^b(\text{mod}^G A) / D^b(\text{proj}^G A),$$

where $G = \mathbb{Z}/a\mathbb{Z}$ for some positive integer a . We denote by $\pi : D^b(\text{mod } A) \rightarrow D_{sg}(\text{mod } A)$ and $\pi^G : D^b(\text{mod}^G A) \rightarrow D_{sg}(\text{mod}^G A)$ the localization functor.

Theorem 2.7 ([7, 23, 24]). *Let A be a positively graded Gorenstein algebra. Then for any group $G = \mathbb{Z}/a\mathbb{Z}$ or \mathbb{Z} , $\text{CM}(A)$ and $\text{CM}^G(A)$ are Frobenius categories with the projective modules and G -graded projective modules as the projective-injective objects respectively. The stable categories $\underline{\text{CM}}(A)$ and $\underline{\text{CM}}^G(A)$ are triangulated equivalent to $D_{sg}(\text{mod } A)$ and $D_{sg}(\text{mod}^G A)$ respectively.*

In the following, we always assume that A is a positively graded algebra.

The additive functor $F_a : \text{mod}^{\mathbb{Z}} A \rightarrow \text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$ induces an additive functor

$$D^b F_a : D^b(\text{mod}^{\mathbb{Z}} A) \rightarrow D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A),$$

which is a triangulated functor. Similar to [52, Proposition 2.25], we get the following lemma.

Lemma 2.8 ([52]). *We have $\text{thick}(\text{Im } D^b F_a) = D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$.*

Proof. The proof is similar to that of [52, Proposition 2.6]. Obviously, $\text{Im } D^b F_a$ contains all simple objects in $\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$. For any $M \in \text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$, it has a finite filtration by simple objects in $\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$. Since short exact sequences in $\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$ gives rise to triangles in $D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$, $\text{thick}(\text{Im } D^b F_a)$ contains all objects in $\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$, and then $\text{thick}(\text{Im } D^b F_a) = D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$. \square

For any $i > 0$, similar to the definition of F_a , we can define an additive functor F_a^{ia} from $\text{mod}^{\mathbb{Z}/ia\mathbb{Z}} A$ to $\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A$ which is also exact. It is easy to see that $F_a = F_a^{ia} \circ F_{ia}$. Then F_a^{ia} also induces a functor $D^b F_a^{ia}$ from $D^b(\text{mod}^{\mathbb{Z}/ia\mathbb{Z}} A)$ to $D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$.

Since all the exact functors F_a and F_a^{ia} preserves projective objects, they induce the triangle functors $\underline{F}_a : D_{sg}(\text{mod}^{\mathbb{Z}} A) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$, and $\underline{F}_a^{ia} : D_{sg}(\text{mod}^{\mathbb{Z}/ia\mathbb{Z}} A) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$ respectively. In particular, we obtain the following commutative diagram:

$$\begin{array}{ccc}
D^b(\text{mod}^{\mathbb{Z}} A) & \xrightarrow{\pi} & D_{sg}(\text{mod}^{\mathbb{Z}} A) \\
\downarrow D^b F_{ia} & & \downarrow \underline{F}_{ia} \\
D^b(\text{mod}^{\mathbb{Z}/ia\mathbb{Z}} A) & \xrightarrow{\pi^{\mathbb{Z}/ia\mathbb{Z}}} & D_{sg}(\text{mod}^{\mathbb{Z}/ia\mathbb{Z}} A) \\
\downarrow D^b F_a^{ia} & & \downarrow \underline{F}_a^{ia} \\
D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A) & \xrightarrow{\pi^{\mathbb{Z}/a\mathbb{Z}}} & D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)
\end{array}
\quad \begin{array}{c} \\ \\ \underline{F}_a \\ \\ \end{array}$$

(Note: The diagram is a commutative square with curved arrows on the sides. The left side has a curved arrow from $D^b(\text{mod}^{\mathbb{Z}} A)$ to $D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$ labeled $D^b F_a$. The right side has a curved arrow from $D_{sg}(\text{mod}^{\mathbb{Z}} A)$ to $D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$ labeled \underline{F}_a .)

It is easy to see that the functors F_a and F_a^{ia} preserve Cohen-Macaulay modules, and then induce triangulated functors on their stable categories, we also denote them by $\underline{F}_a : \underline{\text{CM}}^{\mathbb{Z}} A \rightarrow \underline{\text{CM}}^{\mathbb{Z}/a\mathbb{Z}} A$ and $\underline{F}_a^{ia} : \underline{\text{CM}}^{\mathbb{Z}/ia\mathbb{Z}} A \rightarrow \underline{\text{CM}}^{\mathbb{Z}/a\mathbb{Z}} A$ respectively, without confusions since $\underline{\text{CM}}^{\mathbb{Z}} A \simeq D_{sg}(\text{mod}^{\mathbb{Z}} A)$ and $\underline{\text{CM}}^{\mathbb{Z}/a\mathbb{Z}} A \simeq D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$.

Lemma 2.9. *Let A be a positively graded Gorenstein algebra. Then we have $\underline{\text{CM}}^{\mathbb{Z}/a\mathbb{Z}} A = \text{thick}(\underline{F}_a) = D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$, $\text{thick}(D^b F_a^{ia}) = D^b(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$, and $\underline{\text{CM}}^{\mathbb{Z}/a\mathbb{Z}} A = \text{thick}(\underline{F}_a^{ia}) = D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$.*

Proof. Since the functors $\pi, \pi^{\mathbb{Z}/a\mathbb{Z}}$ and $\pi^{\mathbb{Z}/ia\mathbb{Z}}$ are dense, it follows from Lemma 2.8 and the above commutative diagram immediately. \square

Following [52], the truncation functors

$$(-)_{\geq i} : \text{mod}^{\mathbb{Z}} A \rightarrow \text{mod}^{\mathbb{Z}} A, \quad (-)_{\leq i} : \text{mod}^{\mathbb{Z}} A \rightarrow \text{mod}^{\mathbb{Z}} A$$

are defined as follows. For a \mathbb{Z} -graded A -module X , $X_{\geq i}$ is a \mathbb{Z} -graded sub A -module of X defined by

$$(X_{\geq i})_j := \begin{cases} 0 & \text{if } j < i \\ X_j & \text{if } j \geq i, \end{cases}$$

and $X_{\leq i}$ is a \mathbb{Z} -graded factor A -module $X/X_{\geq i+1}$ of X .

Now we define a \mathbb{Z} -graded A -module by

$$T := \bigoplus_{i \geq 0} A(i)_{\leq 0}.$$

Let \mathcal{T} be a triangulated category. An object $T \in \mathcal{T}$ is called *tilting* if it satisfies the following conditions.

- (1) $\text{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i \neq 0$.
- (2) $\mathcal{T} = \text{thick}_{\mathcal{T}} T$.

Let \mathcal{T} be an algebraic triangulated Krull-Schmidt category. If \mathcal{T} has a tilting object T , then there exists a triangle equivalence $\mathcal{T} \simeq K^b(\text{proj End}_{\mathcal{T}}(T))$, see [28].

Lemma 2.10 ([52, 36]). *Let A be a positively graded Gorenstein algebra where A_0 has finite global dimension. If $T = \bigoplus_{i \geq 0} A(i)_{\leq 0}$ is a Cohen-Macaulay A -module, then T is tilting object in $\underline{\text{CM}}^{\mathbb{Z}}(A)$.*

At the end of this section, we consider the case when $T = \bigoplus_{i \geq 0} A(i)_{\leq 0}$ is a Cohen-Macaulay A -module. Let Γ be the endomorphism of T in $\underline{\mathbf{CM}}^{\mathbb{Z}} A$.

Theorem 2.11 ([52]). *Take a decomposition $T = \underline{T} \oplus P$ in $\text{Mod}^{\mathbb{Z}} A$ where \underline{T} is a direct sum of all indecomposable non-projective direct summand of T . Then*

- (a) *\underline{T} is finitely generated, and is isomorphic to T in $\underline{\mathbf{CM}}^{\mathbb{Z}} A$.*
- (b) *There exists an algebra isomorphism $\Gamma \simeq \text{End}_A(\underline{T})_0$, and if A_0 has finite global dimension, then so does Γ .*

Similar to [52, Section 3.2], we take a positive integer l such that $A = A_{\leq l}$. Let $U := \bigoplus_{i=0}^{l-1} A(i)_{\leq 0}$. Then there exists an algebra isomorphism

$$\text{End}_A(U)_0 \simeq \begin{pmatrix} A_0 & A_1 & \cdots & A_{l-2} & A_{l-1} \\ & A_0 & \cdots & A_{l-3} & A_{l-2} \\ & & \ddots & \vdots & \vdots \\ & & & A_0 & A_1 \\ & & & & A_0 \end{pmatrix}.$$

If we decompose $U = \underline{T} \oplus P'$ for some projective direct summand of U , then

$$\text{End}_A(U)_0 \simeq \begin{pmatrix} \text{End}_A(P')_0 & \text{Hom}_A(\underline{T}, P')_0 \\ 0 & \Gamma \end{pmatrix}.$$

2.3. Derived-orbit categories. In this subsection, first we collect basic facts on DG categories and their derived categories from [28]. After that, we recall the definition of derived-orbit categories defined in [52] which is a special case of DG orbit categories defined in [29].

Definition 2.12 ([28]). *An additive category \mathcal{A} is called a differential graded (DG) category if the following conditions are satisfied.*

- (a) *For any $X, Y \in \mathcal{A}$, the morphism set $\text{Hom}_{\mathcal{A}}(X, Y)$ is a \mathbb{Z} -graded abelian group*

$$\text{Hom}_{\mathcal{A}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}^i(X, Y)$$

such that $gf \in \text{Hom}_{\mathcal{A}}^{i+j}(X, Z)$ for any $f \in \text{Hom}_{\mathcal{A}}^i(X, Y)$ and $g \in \text{Hom}_{\mathcal{A}}^j(Y, Z)$ where $X, Y, Z \in \mathcal{A}$ and $i, j \in \mathbb{Z}$.

- (b) *The morphism set is endowed with a differential d such that the equation*

$$d(gf) = (dg)f + (-1)^j g(df)$$

hold for any $f \in \text{Hom}_{\mathcal{A}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{A}}^j(Y, Z)$ where $X, Y, Z \in \mathcal{A}$ and $j \in \mathbb{Z}$.

For a DG category \mathcal{A} , $H^0(\mathcal{A})$ is the additive category whose objects are the same as \mathcal{A} , and the morphism set from X to Y is $H^0(\text{Hom}_{\mathcal{A}}(X, Y))$.

Let \mathcal{A} and \mathcal{B} be DG categories. An additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a *DG functor* if it preserves the grading and commutes with differentials.

Let \mathcal{A} be an additive category and $C_{dg}(\mathcal{A})$ the category of complexes over \mathcal{A} with the morphism sets and the differential defined as follows: For any two complexes L, M and an integer $n \in \mathbb{Z}$, we define $\text{Hom}_{C_{dg}(\mathcal{A})}(L, M)^n$ to be the set formed by the morphisms $f : L \rightarrow M$ of graded objects of degree n , and $\text{Hom}_{C_{dg}(\mathcal{A})}(L, M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{C_{dg}(\mathcal{A})}(L, M)^n$. The differential is the commutator

$$d(f) = d_M f - (-1)^n f d_L$$

for any $f \in \text{Hom}_{C_{dg}(\mathcal{A})}(L, M)^n$. In this case, $H^0(C_{dg}(\mathcal{A}))$ coincides with the homotopy category $K(\mathcal{A})$ of complexes over \mathcal{A} . Let $C_{dg}^b(\mathcal{A})$ be the full subcategory of $C_{dg}(\mathcal{A})$ formed by all bounded complexes. Then $H^0(C_{dg}(\mathcal{A})) = K^b(\mathcal{A})$.

Let \mathcal{A} be a small DG K -category. A DG \mathcal{A} -module is a DG functor $\mathcal{A}^{op} \rightarrow C_{dg}(\text{mod } K)$. Then DG \mathcal{A} -modules form a DG category which we denote by $C_{dg}(\mathcal{A})$. We denote by $K(\mathcal{A}) := H^0(C_{dg}(\mathcal{A}))$ the homotopy category of DG \mathcal{A} -modules. By formally inverting quasi-isomorphisms in $K(\mathcal{A})$, we obtain the derived category $D(\mathcal{A})$ of DG \mathcal{A} -modules. From [28], $K(\mathcal{A})$ and $D(\mathcal{A})$ are triangulated categories.

We call a DG functor $\mathcal{A} \rightarrow C_{dg}(\mathcal{A})$ sending X to $\text{Hom}_{\mathcal{A}}(-, X)$ the *Yoneda embedding*. In this way, \mathcal{A} is a DG full subcategory of $C_{dg}(\mathcal{A})$, and $H^0(\mathcal{A})$ is a full subcategory of $H^0(C_{dg}(\mathcal{A})) = K(\mathcal{A})$, which is also of $D(\mathcal{A})$.

Definition 2.13 ([29]). *Let \mathcal{A} be a DG category. We call the full subcategory $\text{thick}_{D(\mathcal{A})}(H^0(\mathcal{A}))$ of $D(\mathcal{A})$ the triangulated hull of $H^0(\mathcal{A})$.*

Definition 2.14 ([29]). *Let \mathcal{A} be a DG category, and $F : \mathcal{A} \rightarrow \mathcal{A}$ a DG functor. A DG category \mathcal{A}/F^+ is defined as follows.*

- The objects are the same as \mathcal{A} .
- For $X, Y \in \mathcal{A}$, the morphism set from X to Y is defined by

$$\text{Hom}_{\mathcal{A}/F^+}(X, Y) := \bigoplus_{n=0}^{\infty} \text{Hom}_{\mathcal{A}}(F^n X, Y).$$

- The differential of \mathcal{A}/F^+ is induced from that of \mathcal{A} .

The DG orbit category \mathcal{A}/F is defined with the objects the same as \mathcal{A} , and the morphism set from X to Y is defined by

$$\text{Hom}_{\mathcal{A}/F}(X, Y) := \text{colim}(\text{Hom}_{\mathcal{A}/F^+}(X, Y) \xrightarrow{F} \text{Hom}_{\mathcal{A}/F^+}(X, FY) \xrightarrow{F} \cdots),$$

with the differential of \mathcal{A}/F induced from that of \mathcal{A}/F^+ .

Let \mathcal{A} be a DG category, $F : \mathcal{A} \rightarrow \mathcal{A}$ a DG functor, and the induced functor $H^0(F) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{A})$. It is easy to see that

$$H^0(\mathcal{A}/F) \simeq H^0(\mathcal{A})/H^0(F),$$

and then the triangulated hull $\text{thick}_{D(\mathcal{A}/F)}(H^0(\mathcal{A}/F))$ of $H^0(\mathcal{A}/F)$ is a triangulated category which contains $H^0(\mathcal{A})/H^0(F)$ as a full subcategory, and is generated by $H^0(\mathcal{A})/H^0(F)$, see [52, Proposition 5.8].

Definition 2.15 ([52]). *Let Λ be an algebra of finite global dimension, M a bounded complex of $\Lambda^{op} \otimes_K \Lambda$ -modules. Let $\mathcal{A} := C_{dg}^b(\text{proj } \Lambda)$ be the DG category of bounded complexes over $\text{proj } \Lambda$. Then $H^0(\mathcal{A}) = K^b(\text{proj } \Lambda) = D^b(\text{mod } \Lambda)$. Since $\Lambda^{op} \otimes_K \Lambda$ has finite global dimension, there exists a quasi-isomorphism $pM \rightarrow M$ with a bounded complex pM of projective $\Lambda^{op} \otimes_K \Lambda$ -modules. Consider the DG functor:*

$$F := - \otimes_{\Lambda} pM : \mathcal{A} \rightarrow \mathcal{A}.$$

Then we have

$$H^0(F) = - \otimes_{\Lambda}^{\mathbb{L}} M : D^b(\text{mod } \Lambda) \rightarrow D^b(\text{mod } \Lambda).$$

We call the triangulated hull

$$D(\Lambda, M) := \text{thick}_{D(\mathcal{A}/F)}(H^0(\mathcal{A}/F))$$

of $H^0(\mathcal{A}/F)$ the derived-orbit category of (Λ, M) .

Definition 2.16 ([29, 15]). Let Λ be an algebra of finite global dimension. For $M = \Sigma^2 \Lambda$, and $F = - \otimes_{\Lambda} \Sigma^2 \Lambda$, where Σ is the shift functor, we call the derived-orbit category $D(\Lambda, \Sigma^2 \Lambda)$ of $(\Lambda, \Sigma^2 \Lambda)$ the root category of Λ , and denote it by \mathcal{R}_{Λ} .

It is worth noting that when Λ is a hereditary algebra, Peng and Xiao [42] proved that the m -periodic orbit category $D^b(\text{mod } \Lambda)/\Sigma^m$ is triangulated via the homotopy category of m -periodic complexes of projective Λ -modules for any $m > 1$ (Ringel and Zhang [48] proved that for the case $m = 1$). In fact, in this case, the derived-orbit category $D(\Lambda, \Sigma^m \Lambda)$ of $(\Lambda, \Sigma^m \Lambda)$ is triangulated equivalent to the m -periodic orbit category $D^b(\text{mod } \Lambda)/\Sigma^m$, see [42, 29].

Let A be a positively graded Gorenstein algebra, and Γ an algebra of finite global dimension. Let $U \in D_{sg}(\text{mod}^{\mathbb{Z}}(\Gamma \otimes_K A))$, $N \in D^b(\text{mod}(\Gamma^{op} \otimes_K \Gamma))$, and

$$F := - \otimes_{\Gamma}^{\mathbb{L}} N : D^b(\text{mod } \Gamma) \rightarrow D^b(\text{mod } \Gamma).$$

We consider the derived tensor functor

$$- \otimes_{\Gamma}^{\mathbb{L}} U : D^b(\text{mod } \Gamma) \rightarrow D^b(\text{mod}^{\mathbb{Z}} A)$$

and the canonical functor $\pi : D^b(\text{mod}^{\mathbb{Z}} A) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}} A)$. Composing them, we have a triangulated functor

$$G := \pi \circ (- \otimes_{\Gamma}^{\mathbb{L}} U) : D^b(\text{mod } \Gamma) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}} A).$$

Theorem 2.17. (see [27, Theorem A.20]) We fix an integer a . Assume that there exists a triangle

$$P \rightarrow U(a) \rightarrow N \otimes_{\Gamma}^{\mathbb{L}} U \rightarrow \Sigma P$$

in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma^{op} \otimes_K A))$ such that P belongs to $K^b(\text{proj}^{\mathbb{Z}} A)$ as an object in $D^b(\text{mod}^{\mathbb{Z}} A)$. Then there exists an additive functor $D^b(\text{mod } \Gamma)/F \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}} A)/(a)$ and a triangulated functor $\tilde{G} : D(\Gamma, N) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A)$ which makes the diagram

$$\begin{array}{ccc} D^b(\text{mod } \Gamma) & \xrightarrow{G} & D_{sg}(\text{mod}^{\mathbb{Z}} A) \\ \downarrow & & \downarrow \\ D^b(\text{mod } \Gamma)/F & \xrightarrow{\quad} & D_{sg}(\text{mod}^{\mathbb{Z}} A)/(a) \\ \downarrow & & \downarrow \\ D(\Gamma, N) & \xrightarrow{\tilde{G}} & D_{sg}(\text{mod}^{\mathbb{Z}/a\mathbb{Z}} A) \end{array} \quad \begin{array}{c} \curvearrowright \\ F_a \end{array}$$

commutative.

3. SINGULARITY CATEGORIES OF H^k

Let (Q°, I°) be a bound quiver. Denote by $\Lambda^{\circ} = KQ^{\circ}/I^{\circ}$. We always assume that Λ° has finite global dimension. Let Q be the quiver obtained from Q° by adding one loop ϵ_i for each vertex i in Q° . For any positive integer k , $H^k := H^k(\Lambda^{\circ}) := KQ/I_k$, where I_k is the ideal of KQ defined by the following relations:

(H1) For each vertex i we have the *nilpotent relation*

$$\epsilon_i^k = 0.$$

(H2) For each arrow $(\alpha : i \rightarrow j) \in Q^{\circ}$ we have the *commutativity relation*

$$\epsilon_i \alpha = \alpha \epsilon_j.$$

(H3) $I^{\circ} \subset I_k$ by viewing I° to be in KQ naturally.

From [30], we get that H^k is isomorphic to the algebra $\Lambda^\circ \otimes_K R_k$, where $R_k = K[X]/(X^k)$ is the truncated polynomial ring. So H^k is a finite-dimensional K -algebra and the representations of H^k are nothing else than representations of Q over the ground ring R_k .

This algebra is also considered in [17] for $I^\circ = 0$. In this case, for $k = 1$, $H^1 = \Lambda^\circ$ which is a hereditary algebra; for $k = 2$, it is shown in [48] that H^2 is a 1-Gorenstein algebra, and that the stable category of $\text{CM}(H^2)$ is triangulated equivalent to the orbit category $D^b(\text{mod } \Lambda^\circ)/\Sigma$ of the bounded derived category $D^b(\text{mod } KQ^\circ)$ of the path algebra KQ° modulo the shift functor Σ ; for Q a quiver of type \mathbb{A}_2 , Ringel and Schmidmeier [47] studied the category $\text{CM}(H^k)$ for any $k > 0$.

Similar to the global dimensions of tensor algebras, we get the following lemma.

Lemma 3.1 (see e.g. [39]). *Let B and C be two finite-dimensional Gorenstein algebras over K . Then $B \otimes_K C$ is a Gorenstein algebra. In particular, $\text{G. dim}(B \otimes_K C) = \text{G. dim } B + \text{G. dim } C$.*

Thus H^k is a finite-dimensional Gorenstein algebra with $\text{G. dim } H^k = \text{gl. dim } \Lambda^\circ$. We endow KQ to be a \mathbb{Z} -graded algebra with ϵ_i degree 1 for any vertex i , and α degree 0 for any arrow $(\alpha : i \rightarrow j) \in Q^\circ$. Then obviously, I_k is homogeneous, and H^k is a positively graded algebra. Similarly, H^k is natural a $\mathbb{Z}/m\mathbb{Z}$ -graded algebra for any $m > 1$.

For any $m > 0$, let Q_{C_m} be the quiver defined by the following:

(C1) The set of the vertices of Q_{C_m} is

$$\{i^j | i \text{ is a vertex of } Q^\circ, 0 \leq j < m\}.$$

(C2) The set of the arrows of Q_{C_m} is

$$\{\alpha^j : s(\alpha)^j \rightarrow t(\alpha)^j | \alpha \text{ is an arrow of } Q^\circ, 0 \leq j < m\}$$

$$\bigcup \{\theta_i^j : i^j \rightarrow i^{j+1} | i \text{ is a vertex of } Q^\circ, 0 \leq j < m-1\}.$$

Let $C_m := KQ_{C_m}/J_m$ be the quotient algebra of KQ_{C_m} with the ideal J_m defined by the following relations:

(C3) For each arrow $(\alpha : s(\alpha) \rightarrow t(\alpha)) \in Q^\circ$ we have the *commutativity relation*

$$\alpha^j \theta_{t(\alpha)}^j = \theta_{s(\alpha)}^j \alpha^{j+1} \text{ for any } 0 \leq j < m-1.$$

(C4) For each element $\sum_{i \in \mathcal{I}} a_i \alpha_1 \cdots \alpha_{s_i} \in I^\circ$, we have $\sum_{i \in \mathcal{I}} a_i \alpha_1^j \cdots \alpha_{s_i}^j \in J_m$ for each $0 \leq j < m$.

For any positive integer m , let $Q_{\mathbb{A}_m}$ be the quiver $0 \rightarrow 1 \rightarrow \cdots \rightarrow m-1$ of Dynkin type \mathbb{A}_m . Denote by B_m the hereditary algebra $KQ_{\mathbb{A}_m}$. It is easy to see that $C_m \cong \Lambda^\circ \otimes B_m = T_m(\Lambda^\circ)$, where $T_m(\Lambda^\circ)$ is the upper triangular $m \times m$ matrix algebra with coefficients in Λ° , i.e.

$$T_m(\Lambda^\circ) = \begin{pmatrix} \Lambda^\circ & \Lambda^\circ & \cdots & \Lambda^\circ \\ & \Lambda^\circ & \cdots & \Lambda^\circ \\ & & \ddots & \vdots \\ & & & \Lambda^\circ \end{pmatrix}_{m \times m}.$$

It is well known that C_m is a finite-dimensional algebra of global dimension $1 + \text{gl. dim } \Lambda^\circ$ for $m > 1$, and of global dimension $\text{gl. dim } \Lambda^\circ$ for $m = 1$.

Furthermore, let $Q_{\mathbb{A}_\infty}$ be the unbounded quiver

$$\cdots \xrightarrow{\alpha_{-2}} -1 \xrightarrow{\alpha_{-1}} 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} \cdots.$$

For any $k > 0$, let $\tilde{H}^k := \Lambda^\circ \otimes_K (KQ_{\mathbb{A}_\infty} / \langle \alpha_i \cdots \alpha_{i+k-1} | i \in \mathbb{Z} \rangle)$. Then \tilde{H}^k is a locally finite-dimensional algebra without identity, obviously, it is isomorphic to the following algebra

$$\begin{pmatrix} \ddots & \ddots & & \ddots & & & & 0 \\ & (\tilde{H}^k)_{00} & (\tilde{H}^k)_{01} & \cdots & (\tilde{H}^k)_{0,k-1} & & & \\ & & (\tilde{H}^k)_{11} & (\tilde{H}^k)_{12} & \cdots & (\tilde{H}^k)_{1,k} & & \\ & & & (\tilde{H}^k)_{22} & (\tilde{H}^k)_{23} & \cdots & (\tilde{H}^k)_{2,k+1} & \\ 0 & & & & \ddots & \ddots & & \ddots \end{pmatrix}$$

where matrices have only finitely many non-zero entries, and $(\tilde{H}^k)_{ij} = \Lambda^\circ$ for any $i \in \mathbb{Z}$ and $i \leq j < k+i$, all the remaining entries are zero and the multiplication is induced from that of Λ° . In the following, we always identify them, and denote both of them by \tilde{H}^k . The identity maps $(\tilde{H}^k)_{ij} \rightarrow (\tilde{H}^k)_{i+1,j+1}$ induce an automorphism φ of \tilde{H}^k . The orbit category $\tilde{H}^k / \langle \varphi \rangle$ inherits from \tilde{H}^k the structure of a K -algebra and easily seen to be isomorphic to H^k . The projection functor $G : \tilde{H}^k \rightarrow H^k$ is thus a *Galois covering* with infinite cyclic group generated by φ , see [16]. We denote by $G_\lambda : \text{mod } \tilde{H}^k \rightarrow \text{mod } H^k$ the associated *push-down functor*.

Definition 3.2 ([9, 5]). *For any group G , and R a G -graded algebra, we construct an algebra, which is called the smash product of R with G as follows: let $R \# G^*$ be the free left R -module with basis $\{p_g | g \in G\}$, and with multiplication given by the following: elements rp_g, sp_h multiply by $(rp_g)(sp_h) = rs_{gh^{-1}}p_h$ and multiplication of sums of such elements is then defined by linearity.*

From [9, Proposition 1.4], in case of G being finite, $R \# G^*$ is a free left and right R -module with basis $\{p_g | g \in G\}$, a set of orthogonal idempotents whose sum is 1.

Let $R = \bigoplus_{g \in G} R_g$ be a G -graded algebra with identity 1 and with a complete set of orthogonal idempotents $e_i, i = 1, \dots, n$, and with $e_i \in R_e$, where $e \in G$ is the unit. There is an isomorphic functor from $\text{Mod } R \# G^*$ to $\text{Mod}^G R$, whose restriction on $\text{mod } R \# G^*$ is again an isomorphic functor from $\text{mod } R \# G^*$ to $\text{mod}^G R$, see [9, 5, 34, 1].

Let $F; \text{mod}^G R \rightarrow \text{mod } R$ be the forgetful functor. We use the notation $\text{mod}_\infty(R)$ for $F(\text{mod}^G R)$, the full subcategory of $\text{mod } R$ with objects isomorphic to R -modules of the form $F(X)$ for some $X \in \text{mod}^G R$. Objects of $\text{mod}_\infty(R)$ are said to be *gradable*.

Lemma 3.3. *For any $k > 0$, we get that $\text{mod } \tilde{H}^k$ is equivalent to $\text{mod}^{\mathbb{Z}} H^k$.*

Proof. It is easy to check that \tilde{H}^k is isomorphic to the smash product $H^k \# \mathbb{Z}^*$, which implies the result immediately. \square

Let A be a positively graded Gorenstein algebra. We say that A has *Gorenstein parameter* p if $\text{soc } A$ is contained in A_p .

Lemma 3.4. *We have that $H^k = \bigoplus_{i=0}^\infty (H^k)_i = \bigoplus_{i=0}^{k-1} (H^k)_i$ is a positively graded Gorenstein algebra of Gorenstein parameter $k-1$ with the property that $(H^k)_i \cong \Lambda^\circ$ for any $k > 0$ and $0 \leq i \leq k-1$.*

Proof. We only need to prove that H^k has Gorenstein parameter $k-1$, which follows from its definition easily. \square

Lemma 3.5. *Let $\underline{T} = \bigoplus_{i=0}^{k-2} H^k(i)_{\leq 0}$ for any $k \geq 2$. Then \underline{T} is a tilting object in $D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$, and its endomorphism*

$$\Gamma_k := \text{Hom}_{D_{sg}(\text{mod}^{\mathbb{Z}} H^k)}(\underline{T})_0$$

is isomorphic to $T_{k-1}(\Lambda^\circ)$. In particular, $G = \pi \circ (-\otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}) : D^b(\text{mod } T_{k-1}(\Lambda^\circ)) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$ is a triangle equivalence.

Proof. The proof is similar to that in [52, Section 6.1]. Note that $H^k(i)_{\leq 0}$ is a projective H^k -module for any $i \geq k-1$ since H^k has Gorenstein parameter $k-1$. So $T = \bigoplus_{i=0}^{\infty} H^k(i)_{\leq 0}$ is isomorphic to $\underline{T} = \bigoplus_{i=0}^{k-2} H^k(i)_{\leq 0}$ in $D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$. Since $\Gamma_k := \text{Hom}_{D_{sg}(\text{mod}^{\mathbb{Z}} H^k)}(\underline{T})_0$ which is viewed as a \mathbb{Z} -graded algebra concentrated at degree zero, we regard $\Gamma_k^{op} \otimes_K H^k$ as a \mathbb{Z} -graded algebra naturally, and regard \underline{T} as a \mathbb{Z} -graded $\Gamma_k^{op} \otimes_K H^k$ -module. For convenience, we write \underline{T} as the following matrix form,

$$\underline{T} = \begin{pmatrix} H^k(k-2)_{\leq 0} \\ H^k(k-3)_{\leq 0} \\ \vdots \\ H^k(1)_{\leq 0} \\ H^k(0)_{\leq 0} \end{pmatrix} = \begin{pmatrix} 2-k & 3-k & \cdots & -1 & 0 \\ \Lambda^\circ & \Lambda^\circ & \cdots & \Lambda^\circ & \Lambda^\circ \\ & \Lambda^\circ & \cdots & \Lambda^\circ & \Lambda^\circ \\ & & \ddots & \vdots & \vdots \\ & & & \Lambda^\circ & \Lambda^\circ \\ & & & & \Lambda^\circ \end{pmatrix},$$

where $2-k, 3-k, \dots, -1, 0$ denote the degrees. It is easy to see that Γ_k is isomorphic to $T_{k-1}(\Lambda^\circ)$. The action of Γ_k on \underline{T} from the left is given by the matrix multiplication. Thus \underline{T} is isomorphic to Γ_k as a Γ_k^{op} -module. Similarly, we get that

$$H^k(k-1)_{\leq 0} = \begin{pmatrix} 1-k & 2-k & 3-k & \cdots & -1 & 0 \\ \Lambda^\circ & \Lambda^\circ & \Lambda^\circ & \cdots & \Lambda^\circ & \Lambda^\circ \end{pmatrix},$$

it is easy to see that $H^k(i)_{\leq 0}$ is a submodule of $H^k(k-1)_{\leq 0}$ for any $0 \leq i \leq k-2$. Since H^k is a positively graded Gorenstein algebra, we get that \underline{T} is a graded Cohen-Macaulay H^k -module. Then Theorem 2.10 yields that \underline{T} is a tilting object in $D_{sg}(\text{mod}^{\mathbb{Z}} H^k)$. \square

Remark 3.6. For $N > 0$, it is worth noting that O. Iyama, K. Kato and Jun-ichi Miyachi defined the homotopy category of N -complexes and the derived category of N -complexes in [26]. The homotopy category of N -complexes of projective Λ° -modules is triangle equivalent to the homotopy category of projective $T_{N-1}(\Lambda^\circ)$ -modules [26, 3].

Using [49, Proposition 1.2], one can prove that $\text{CM}^{\mathbb{Z}}(H^k)$ is equivalent to the category of k -complex of projective Λ° -modules as Frobenius categories, which yields that $\underline{\text{CM}}^{\mathbb{Z}}(H^k)$ is triangle equivalent to the homotopy category of k -complexes of projective Λ° -modules, and so Lemma 3.5 can be proved in this way.

Let

$$M = \bigoplus_{i=2k-2}^k H^k(i)_{\geq 1-k}.$$

Similarly, it is convenient to write M as the following matrix form

$$M = \begin{pmatrix} H^k(2k-2)_{\geq 1-k} \\ H^k(2k-1)_{\geq 1-k} \\ \vdots \\ H^k(1-k)_{\geq 1-k} \\ H^k(k)_{\geq 1-k} \end{pmatrix} = \begin{pmatrix} 1-k & 2-k & \cdots & -2 & -1 \\ \Lambda^\circ & \Lambda^\circ & & & \\ \Lambda^\circ & \Lambda^\circ & & & \\ \vdots & \vdots & \ddots & & \\ \Lambda^\circ & \Lambda^\circ & \cdots & \Lambda^\circ & \\ \Lambda^\circ & \Lambda^\circ & \cdots & \Lambda^\circ & \Lambda^\circ \end{pmatrix}.$$

The algebra Γ_k acts on M from both sides by matrix multiplication, which implies that M is a $\Gamma_k^{op} \otimes_K \Gamma_k$ -module. Since the action of Γ_k on M from the left commutes with that of H^k from the right, we can regard M as a \mathbb{Z} -graded $\Gamma_k^{op} \otimes H^k$ -module, see [52, Section 6].

Lemma 3.7. *The following assertions hold.*

- (a) *There exist isomorphisms $M \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} \simeq M$ and $\underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} \cong \underline{T}$ in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes_K H^k))$.*
- (b) *There exist two short exact sequences*

$$0 \rightarrow M \rightarrow \bigoplus_{i=k}^{2k-2} H^k(i) \rightarrow \underline{T}(k) \rightarrow 0, \text{ and } 0 \rightarrow \underline{T} \rightarrow (H^k(k-1))^{\oplus(k-1)} \rightarrow M \rightarrow 0$$

in $\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes H^k)$.

Proof. Similar to the proof of [52, Lemma 6.3 and Lemma 6.4 (1)], we can prove (a) and get the first exact sequence in (b). So we only need to prove that there exists the second exact sequence in (b). It is easy to see that it is an exact sequence in $\text{mod}^{\mathbb{Z}} H^k$. Note that the left Γ_k -module structure of $(H^k(k-1))^{\oplus(k-1)}$ is induced by the matrix multiplication. Then $(H^k(k-1))^{\oplus(k-1)}$ is a $(\Gamma_k^{op} \otimes H^k)$ -module. Together with the left Γ_k -module structures of \underline{T} and M , it is routine to check that

$$0 \rightarrow \underline{T} \rightarrow (H^k(k-1))^{\oplus(k-1)} \rightarrow M \rightarrow 0$$

is an exact sequence in $\text{mod}^{\mathbb{Z}} \Gamma_k^{op}$, and then in $\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes H^k)$. \square

Since each short exact sequence in $\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes A)$ gives a triangle in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes_K H^k))$, so we get the following two triangles in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes_K H^k))$:

$$\bigoplus_{i=k}^{2k-2} H^k(i) \rightarrow \underline{T}(k) \rightarrow \Sigma M \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} \rightarrow \Sigma \left(\bigoplus_{i=k}^{2k-2} H^k(i) \right)$$

and

$$\Sigma((H^k(k-1))^{\oplus(k-1)}) \rightarrow \Sigma M \rightarrow \Sigma^2 \underline{T} \rightarrow \Sigma^2((H^k(k-1))^{\oplus(k-1)}).$$

Combining them, we get the following commutative diagram by octahedral axiom:

$$\begin{array}{ccccccc} \bigoplus_{i=k}^{2k-2} H^k(i) & \longrightarrow & V & \longrightarrow & \Sigma((H^k(k-1))^{\oplus(k-1)}) \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} & \longrightarrow & \Sigma(\bigoplus_{i=k}^{2k-2} H^k(i)) \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ \bigoplus_{i=k}^{2k-2} H^k(i) & \longrightarrow & \underline{T}(k) & \longrightarrow & \Sigma M \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} & \longrightarrow & \Sigma(\bigoplus_{i=k}^{2k-2} H^k(i)) \\ & & \downarrow & & \downarrow & & \\ & & \Sigma^2 \underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} & \xlongequal{\quad} & \Sigma^2 \underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} & & \\ & & \downarrow & & \downarrow & & \\ & & \Sigma V & \longrightarrow & \Sigma^2((H^k(k-1))^{\oplus(k-1)}) \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} & & \end{array}$$

So we get a triangle in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma_k^{op} \otimes_K H^k))$:

$$(1) \quad V \xrightarrow{\alpha} \underline{T}(k) \xrightarrow{\beta} \Sigma^2 \underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} \xrightarrow{\gamma} \Sigma V.$$

Similar to Lemma 3.7 (1), we get that $(H^k(k-1))^{\oplus(k-1)} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T} \simeq (H^k(k-1))^{\oplus(k-1)}$, which yields that $V \in K^b(\text{proj}^{\mathbb{Z}} H^k)$.

Proposition 3.8. *We have the following commutative diagram of triangle equivalences up to an isomorphism of functors.*

$$\begin{array}{ccc} D^b(\text{mod } \Gamma_k) & \xrightarrow{G} & D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k) \\ \downarrow F & & \downarrow (k) \\ D^b(\text{mod } \Gamma_k) & \xrightarrow{G} & D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k), \end{array}$$

where

$$F := - \otimes_{\Gamma_k}^{\mathbb{L}} \Sigma^2 \underline{T} : D^b(\text{mod } \Gamma_k) \rightarrow D^b(\text{mod } \Gamma_k)$$

which is isomorphic to the shift functor $\Sigma^2 = - \otimes_{\Gamma_k}^{\mathbb{L}} \Sigma^2 \Gamma_k$ and

$$G = \pi \circ (- \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}) : D^b(\text{mod } \Gamma_k) \rightarrow D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k).$$

In particular, there is an isomorphism of functors $(k) \circ G \rightarrow G \circ \Sigma^2$.

Proof. The proof is similar to that of [52, Proposition 6.1], we give it here for convenience.

For $X \in D^b(\text{mod } \Gamma_k)$, we have a triangle

$$\pi(X \otimes_{\Gamma_k}^{\mathbb{L}} V) \rightarrow \pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}(k)) \xrightarrow{\pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \beta)} \pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \Sigma^2 \underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}) \rightarrow \Sigma \pi(X \otimes_{\Gamma_k}^{\mathbb{L}} V)$$

in $D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k)$ by applying $\pi \circ (X \otimes_{\Gamma_k}^{\mathbb{L}} -)$ to the triangle (1). Since $\pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}(k)) = G(X)(k)$, $\pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \Sigma^2 \underline{T} \otimes_{\Gamma_k}^{\mathbb{L}} \underline{T}) = GF(X)$, and $X \otimes_{\Gamma_k}^{\mathbb{L}} V \in K^b(\text{proj } {}^{\mathbb{Z}} H^k)$, we have an isomorphism

$$\pi(X \otimes_{\Gamma_k}^{\mathbb{L}} \beta) : G(X)(k) \rightarrow GF(X)$$

in $D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k)$. Thus we have an equivalence of functors

$$\pi(- \otimes_{\Gamma_k}^{\mathbb{L}} \beta) : (k) \circ G \rightarrow G \circ F.$$

Note that \underline{T} is isomorphic to Γ_k as a $\Gamma_k^{op} \otimes_K \Gamma_k$ -module, it is easy to see that $F \simeq \Sigma^2$. In particular, we get that an isomorphism

$$(k) \circ G \rightarrow G \circ \Sigma^2$$

of functors. □

Corollary 3.9. (a) *We have an equivalence*

$$D^b(\text{mod } \Gamma_k)/\Sigma^2 \simeq (D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k))/(k) \simeq (\underline{\text{CM}}^{\mathbb{Z}} H^k)/(k).$$

(b) *The root category \mathcal{R}_{Γ_k} of Γ_k is a Hom-finite Krull-Schmidt category.*

Proof. (a) By Proposition 3.8, G induces an equivalence $D^b(\text{mod } \Gamma_k)/\Sigma^2 \simeq (D_{sg}(\text{mod } {}^{\mathbb{Z}} \Gamma_k))/(k) \simeq (\underline{\text{CM}}^{\mathbb{Z}} \Gamma_k)/(k)$.

(b) It is the same to that of [52, Proposition 6.5], we omit it here. □

Theorem 3.10. *For any $k > 0$, $n > 0$, there exists a triangle equivalence*

$$\tilde{G} : D(\Gamma_k, \Sigma^{2n} \Gamma_k) \rightarrow D_{sg}(\text{mod } {}^{\mathbb{Z}/kn\mathbb{Z}} H^k)$$

which makes the following diagram commutative:

$$\begin{array}{ccccc} D^b(\text{mod } \Gamma_k) & \xrightarrow{G} & D_{sg}(\text{mod } {}^{\mathbb{Z}} H^k) & \xrightarrow{\sim} & \underline{\text{CM}}^{\mathbb{Z}} \Gamma_k \\ \downarrow \text{nat.} & & \downarrow F_{kn} & & \downarrow F_{kn} \\ D(\Gamma_k, \Sigma^{2n} \Gamma_k) & \xrightarrow{\tilde{G}} & D_{sg}(\text{mod } {}^{\mathbb{Z}/kn\mathbb{Z}} H^k) & \xrightarrow{\sim} & \underline{\text{CM}}^{\mathbb{Z}/kn\mathbb{Z}} \Gamma_k. \end{array}$$

Proof. The proof is the same to that of [52, Theorem 6.2], we omit it here. \square

We have the following corollary directly.

Corollary 3.11. *For any integer $k > 0$, there exists a triangle equivalence*

$$\tilde{G} : \mathcal{R}_{\Gamma_k} \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}/kn\mathbb{Z}} H^k)$$

which makes the following diagram commutative:

$$\begin{array}{ccccc} D^b(\text{mod } \Gamma_k) & \xrightarrow{G} & D_{sg}(\text{mod}^{\mathbb{Z}} H^k) & \xrightarrow{\sim} & \underline{\text{CM}}^{\mathbb{Z}} \Gamma_k \\ \downarrow \text{nat.} & & \downarrow \underline{F_k} & & \downarrow \underline{F_k} \\ \mathcal{R}_{\Gamma_k} & \xrightarrow{\tilde{G}} & D_{sg}(\text{mod}^{\mathbb{Z}/k\mathbb{Z}} H^k) & \xrightarrow{\sim} & \underline{\text{CM}}^{\mathbb{Z}/k\mathbb{Z}} \Gamma_k. \end{array}$$

The following corollary is a generalization of [48, Theorem 1], which states that $D_{sg}(H^2) \simeq D^b(\text{mod } \Lambda^\circ)/\Sigma$ if Λ° is a hereditary algebra.

Theorem 3.12. *We have a triangle equivalence:*

$$D_{sg}(\text{mod}^{\mathbb{Z}/n\mathbb{Z}} H^2) \simeq D(\Lambda^\circ, \Sigma^n \Lambda^\circ), \text{ for any } n > 0.$$

If Λ° is hereditary or derived equivalent to a hereditary category, then

$$D_{sg}(\text{mod}^{\mathbb{Z}/n\mathbb{Z}} H^2) \simeq D(\Lambda^\circ)/\Sigma^n, \text{ for any } n > 0.$$

In particular, if $\Lambda^\circ = KQ^\circ$, then $D_{sg}(H^2) \simeq D^b(\text{mod } \Lambda^\circ)/\Sigma$.

Proof. For $k = 2$, we get that $\underline{T} = \Lambda^0$, and $\Gamma_2 = \Lambda^0$. So there exists an exact sequence in $\text{mod}^{\mathbb{Z}}((\Lambda^\circ)^{op} \otimes_K H^2)$:

$$0 \rightarrow \Lambda^0 = \underline{T} \rightarrow H^2(1) \rightarrow \underline{T}(1) = \Lambda^0(1) \rightarrow 0.$$

Then there is a triangle

$$H^2(1) \rightarrow \underline{T}(1) \rightarrow \Sigma \underline{T} \rightarrow \Sigma H^2(1)$$

in $D^b(\text{mod}^{\mathbb{Z}}(\Gamma_2^{op} \otimes_K H^2))$. Similar to the proof of Proposition 3.8, there exists a commutative diagram

$$\begin{array}{ccc} D^b(\text{mod } \Lambda^\circ) & \xrightarrow{G} & D_{sg}(\text{mod}^{\mathbb{Z}} H^2) \\ \downarrow \Sigma & & \downarrow (1) \\ D^b(\text{mod } \Lambda^\circ) & \xrightarrow{G} & D_{sg}(\text{mod}^{\mathbb{Z}} H^2), \end{array}$$

i.e., $G\Sigma = (1)G$. Furthermore, for any $n > 0$, $G\Sigma^n = (n)G$. Also similar to the proof of Theorem 3.10, we get that $D^b(\text{mod } \Lambda^\circ)/\Sigma^n \simeq D_{sg}(\text{mod}^{\mathbb{Z}} H^2)/(n)$ and there exists a triangle equivalence

$$\tilde{G} : D(\Lambda^\circ, \Sigma^n \Lambda^\circ) \rightarrow D_{sg}(\text{mod}^{\mathbb{Z}/n\mathbb{Z}} H^2),$$

for any $n > 0$.

If Λ° is hereditary or derived equivalent to a hereditary category, then the derived-orbit category $D(\Lambda^\circ, \Sigma^n \Lambda^\circ)$ is the triangulated hull of $D^b(\text{mod } \Lambda^\circ)/\Sigma^n$ in $D(C_{dg}^b(\text{proj } \Lambda^\circ)/\Sigma^n)$, by the main Theorem in [29], we get that $D^b(\text{mod } \Lambda^\circ)/\Sigma^n$ is already a triangulated category, which implies that $D(\Lambda^\circ, \Sigma^n \Lambda^\circ)$ is triangle equivalent to $D^b(\text{mod } \Lambda^\circ)/\Sigma^n$. In particular, for $n = 1$, we have $\text{mod}^{\mathbb{Z}/1\mathbb{Z}} H^2 \simeq \text{mod } H^2$, and then $D_{sg}(H^2) \simeq D^b(\text{mod } \Lambda^\circ)/\Sigma$. \square

For an additive category \mathcal{A} , let $C^b(\mathcal{A})$ be the category of bounded complexes in \mathcal{A} . Let $K^b(\mathcal{A})$ be the corresponding homotopy category of $C^b(\mathcal{A})$ and let $P = P_{\mathcal{A}}$ denote the natural functor $C^b(\mathcal{A}) \rightarrow K^b(\mathcal{A})$.

For each $m \geq 1$, an $\mathbb{Z}/m\mathbb{Z}$ -graded complex $M^\cdot = (M^i, d^i)_{i \in \mathbb{Z}/m\mathbb{Z}}$ over \mathcal{A} consists of objects M^i in \mathcal{A} and morphisms $d^i : M^i \rightarrow M^{i+1}$ for $i \in \mathbb{Z}/m\mathbb{Z}$ satisfying $d^{i+1}d^i = 0$. A morphism f between two $\mathbb{Z}/m\mathbb{Z}$ -graded complexes $M^\cdot = (M^i, d^i)$ and $N^\cdot = (N^i, c^i)$ is given by a family of morphisms $f^i : M^i \rightarrow N^i$ satisfying $f^{i+1}d^i = c^i f^i$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. The category of $\mathbb{Z}/m\mathbb{Z}$ -graded complexes over \mathcal{A} is denoted by $C_m(\mathcal{A})$. Finally, let $K_m(\mathcal{A})$ be the corresponding homotopy category of $C_m(\mathcal{A})$ and $P_m = P_{m, \mathcal{A}} : C_m(\mathcal{A}) \rightarrow K_m(\mathcal{A})$ be the natural functor.

For each $m \geq 1$, there exists a functor

$$\mathcal{F}^m : C^b(\mathcal{A}) \rightarrow C_m(\mathcal{A})$$

taking $M^\cdot = (M^s, d^s)_{s \in \mathbb{Z}}$ to $\mathcal{F}^m(M^\cdot) = (X^i, d^i)_{i \in \mathbb{Z}/m\mathbb{Z}}$ with

$$X^i = \bigoplus_{s \in i\mathbb{Z}} M^s \text{ and } d^i = \text{diag}\{d^s | s \in i\mathbb{Z}\}.$$

Lemma 3.13 ([53, 42, 48]). *The category $K_m(\text{proj } \Lambda^\circ)$ is triangle equivalent to $D(\Lambda^\circ, \Sigma^m \Lambda^\circ)$ for any $m > 0$. In particular, if Λ° is hereditary or derived equivalent to a hereditary category, then $K_m(\text{proj } \Lambda^\circ)$ is triangle equivalent to $D^b(\text{mod } \Lambda^\circ)/\Sigma^m$ for any $m > 0$.*

Let $\sigma : C^b(\text{proj } \Lambda^\circ) \rightarrow C^b(\text{proj } \Lambda^\circ)$ denote the functor such that for $M^\cdot = (M^i, d^i) \in C_0(\text{proj } \Lambda^\circ)$, $\sigma(M^\cdot) = (X^i, f^i)$ is defined by $X^i = M^{m+i}$ and $f^i = d^{m+i}$ for all $i \in \mathbb{Z}$. Then for each $m \geq 1$, the functor

$$\mathcal{F}^m : C^b(\text{proj } \Lambda^\circ) \rightarrow C_m(\text{proj } \Lambda^\circ)$$

is a Galois G -covering in the sense of [4, Definiton 2.8], where G denotes the infinite cyclic group generated by σ , and the induced functor

$$\mathcal{F}^m : K^b(\text{proj } \Lambda^\circ) \rightarrow K_m(\text{proj } \Lambda^\circ)$$

is also a Galois G -covering, see [42, 8].

By identifying $\text{mod } {}^{\mathbb{Z}}H^2$ with $\text{mod } \tilde{H}^2$, it is obvious that there exists an exact functor Φ^0 from $C^b(\text{mod } \Lambda^\circ)$ to $\text{mod } \tilde{H}^2 = \text{mod } {}^{\mathbb{Z}}H^2$. Furthermore, Φ^0 induces a triangle functor from $D^b(\text{mod } \Lambda^\circ)$ to $D^b(\text{mod } {}^{\mathbb{Z}}H^2)$, which is the natural embedding, we also denote it by Φ^0 .

Let $\tilde{H}_m^2 = \Lambda^\circ \otimes_K KQ_m / \langle \alpha_i \alpha_{i-1} | i \in \mathbb{Z}/m\mathbb{Z} \rangle$, where Q_m is the cyclic quiver with vertices labeled by $\mathbb{Z}/m\mathbb{Z}$ and one arrow from $i-1$ to i for each $i \in \mathbb{Z}/m\mathbb{Z}$. Similarly, $\tilde{H}_m^2 \cong H^2 \# (\mathbb{Z}/m\mathbb{Z})^*$, and it is obvious that there exists an exact functor Φ^m from $C_m(\text{mod } \Lambda^\circ)$ to $\text{mod } \tilde{H}_m^2 = \text{mod } {}^{\mathbb{Z}/m\mathbb{Z}}H^2$.

D. Shen constructs an equivalence of exact categories $\Phi^m : C_m(\text{proj } \Lambda^\circ) \simeq \text{CM}^{\mathbb{Z}/m\mathbb{Z}}(H^2)$ in [49] by describing $\mathbb{Z}/m\mathbb{Z}$ -graded Cohen-Macaulay H^2 -modules clearly, see the following lemma.

Lemma 3.14 ([49]). *Keep the notations as above. Then*

(a) *There are equivalences of exact categories*

$$\Phi^0 : C^b(\text{proj } \Lambda^\circ) \simeq \text{CM}^{\mathbb{Z}}(H^2) \text{ and } \Phi^m : C_m(\text{proj } \Lambda^\circ) \simeq \text{CM}^{\mathbb{Z}/m\mathbb{Z}}(H^2)$$

for any $m > 0$.

(b) *There are equivalences of triangulated categories*

$$\underline{\Phi}^0 : K^b(\text{proj } \Lambda^\circ) \simeq \underline{\text{CM}}^{\mathbb{Z}}(H^2) \text{ and } \underline{\Phi}^m : K_m(\text{proj } \Lambda^\circ) \simeq \underline{\text{CM}}^{\mathbb{Z}/m\mathbb{Z}}(H^2)$$

for any $m > 0$.

The following proposition is a generalization of [48, Theorem 2].

Proposition 3.15. *There are equivalences of exact categories*

$$\Phi^0 : C^b(\text{proj } \Lambda^\circ) \simeq \text{CM}^\mathbb{Z}(H^2) \text{ and } \Phi^m : C_m(\text{proj } \Lambda^\circ) \simeq \text{CM}^{\mathbb{Z}/m\mathbb{Z}}(H^2)$$

for any $m > 0$, which make the following diagram commutative:

$$\begin{array}{ccccccccc} D^b(\text{mod } \Lambda^\circ) & \xleftarrow{\sim} & K^b(\text{proj } \Lambda^\circ) & \xleftarrow{P} & C^b(\text{proj } \Lambda^\circ) & \xrightarrow{\mathcal{F}^m} & C_m(\text{proj } \Lambda^\circ) & \xrightarrow{P_m} & K_m(\text{proj } \Lambda^\circ) & \xrightarrow{\sim} & D(\Lambda^\circ, \Sigma^m \Lambda^\circ) \\ \downarrow G & & \downarrow \Phi^0 & & \downarrow \Phi^0 & & \downarrow \Phi^m & & \downarrow \Phi^m & & \downarrow \tilde{G} \\ D_{sg}(\text{mod }^\mathbb{Z} H^2) & \xleftarrow{\sim} & \underline{\text{CM}}^\mathbb{Z}(H^2) & \xleftarrow{\pi} & \text{CM}^\mathbb{Z}(H^2) & \xrightarrow{F^m} & \text{CM}^{\mathbb{Z}/m\mathbb{Z}}(H^2) & \xrightarrow{\pi^{\mathbb{Z}/m\mathbb{Z}}} & \underline{\text{CM}}^{\mathbb{Z}/m\mathbb{Z}}(H^2) & \xrightarrow{\sim} & D_{sg}(\text{mod }^{\mathbb{Z}/m\mathbb{Z}} H^2). \end{array}$$

Proof. For H^2 , the tilting object \underline{T} we constructed above is isomorphic to Λ° as left Λ° -modules, so the functor $-\otimes_{\Lambda^\circ}^{\mathbb{L}} \underline{T}$ is the natural embedding which maps a complex to \mathbb{Z} -graded H^2 -module, i.e., it coincides with the functor $\Phi^0 : D^b(\text{mod } \Lambda^\circ) \rightarrow D^b(\text{mod }^\mathbb{Z} H^2)$. So $G = \pi \circ (-\otimes_{\Lambda^\circ}^{\mathbb{L}} \underline{T})$ is equivalent to the combination of the natural embedding $D^b(\text{mod } \Lambda^\circ) \rightarrow D^b(\text{mod }^\mathbb{Z} H^2)$ and the projection $\pi : D^b(\text{mod }^\mathbb{Z} H^2) \rightarrow D_{sg}(\text{mod }^\mathbb{Z} H^2)$. Let $\Psi : D_{sg}(\text{mod }^\mathbb{Z} H^2) \rightarrow \underline{\text{CM}}^\mathbb{Z}(H^2)$ be the inverse of the canonical equivalence $\underline{\text{CM}}^\mathbb{Z}(H^2) \rightarrow D_{sg}(\text{mod }^\mathbb{Z} H^2)$. Since Φ^0 induces an exact equivalence between $C^b(\text{proj } \Lambda^\circ)$ and $\text{CM}^\mathbb{Z}(H^2)$, the combination

$$K^b(\text{proj } \Lambda^\circ) \rightarrow D^b(\text{mod } \Lambda^\circ) \xrightarrow{\Phi^0} D^b(\text{mod }^\mathbb{Z} H^2) \xrightarrow{\pi} D_{sg}(\text{mod }^\mathbb{Z} H^2) \xrightarrow{\Psi} \underline{\text{CM}}^\mathbb{Z}(H^2)$$

is equivalent to $\Phi^0 : K^b(\text{proj } \Lambda^\circ) \rightarrow \underline{\text{CM}}^\mathbb{Z}(H^2)$. So the leftmost square is commutative.

The commutativity of the second, third and forth squares follows from the definitions directly.

For the rightmost square, since both $K_m(\text{proj } \Lambda^\circ)$ and $D(\Lambda^\circ, \Sigma^m \Lambda^\circ)$ are generated by stalk complexes by [53, Proposition 2.8, Theorem 2.10], we have $K_m(\text{proj } \Lambda^\circ) = \text{thick}(K^b(\text{proj } \Lambda^\circ)/\Sigma^m)$ and $D(\Lambda^\circ, \Sigma^m \Lambda^\circ) = \text{thick}(D^b(\text{mod } \Lambda^\circ)/\Sigma^m)$. Note that the orbit categories $K^b(\text{proj } \Lambda^\circ)/\Sigma^m$ and $D^b(\text{mod } \Lambda^\circ)/\Sigma^m$ are equivalent, and it induces the equivalence from $K_m(\text{proj } \Lambda^\circ)$ to $D(\Lambda^\circ, \Sigma^m \Lambda^\circ)$. So in order to prove the commutativity of the rightmost square, we only need to check that by restricting it to $K(\text{proj } \Lambda^\circ)/\Sigma^m$, which is similar to the proof of the leftmost square, we omit it here. \square

In [6], Bridgeland defines a localization of the Ringel-Hall algebra $\mathcal{H}(\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\text{proj } \Lambda^\circ))$ with respect to the acyclic complexes, which is called Bridgeland's Ringel-Hall algebra by others, to realize the corresponding quantum group. From above, we get that $\text{CM}^{\mathbb{Z}/2\mathbb{Z}}(H^2) \simeq \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\text{proj } \Lambda^\circ)$ is equivalent as exact categories. So one can use the localization of the Ringel-Hall algebra $\mathcal{H}(\text{CM}^{\mathbb{Z}/2\mathbb{Z}}(H^2))$ with respect to the projective objects, which is called the semi-derived Hall algebra of $\text{CM}^{\mathbb{Z}/2\mathbb{Z}}(H^2)$ in [20], to realize the corresponding quantum group.

4. CLASSIFICATION OF H^k UP TO COHEN-MACAULAY MODULE TYPES FOR Λ° HEREDITARY

In this section, we always assume that $\Lambda^\circ = KQ^\circ$, where Q° is a connected acyclic quiver, and K is an algebraically closed field with its characteristic zero.

Following Section 3, for any G -graded algebra A , let $F : \text{mod } A\#G^* \rightarrow \text{mod }^G A$ be the isomorphic functor. Since this isomorphic functor F preserves projective modules, injective modules, we get that for any G -graded K -algebra A , $A\#G^*$ is Gorenstein if and only if A is (graded) Gorenstein. Similarly, F preserves Cohen-Macaulay modules, analogous to the graded representation type of G -graded K -algebras defined in [21], it is reasonable to define the graded Cohen-Macaulay type of G -graded K -algebras as follows:

Definition 4.1. *For a finite group G , a finite-dimensional G -graded K -algebra A is graded CM-finite (resp. graded CM-tame, graded CM-wild) if the smash product $A\#G^*$ is CM-finite (resp. CM-tame, CM-wild).*

Before going on, we fix a notation at first. Let A be a K -algebra and M, N two A -modules. If M is a direct summand of N as A -modules, then we denote it by $M|N$.

Lemma 4.2 ([33]). *Let G be a finite group, and A be a G -graded algebra. Then for any finitely generated $A\#G^*$ -module X , $X|X \otimes_A (A\#G^*)$.*

Similar to [33], we get the following result. Note that in our setting, K is of characteristic zero.

Theorem 4.3. *For a finite group G , a finite-dimensional G -graded Gorenstein K -algebra A is graded CM-finite (resp. graded CM-tame) if and only if (forgotten G -grading) A is CM-finite (resp. CM-tame). Furthermore, if A is graded CM-wild, then A is CM-semi-wild.*

Proof. The proof given here is based on that of [33, Proposition 4.2, Proposition 4.4].

If A is CM-finite, let $\{B_1, \dots, B_t\}$ be a complete set of non-isomorphic indecomposable Cohen-Macaulay A -modules. Suppose X is an indecomposable Cohen-Macaulay $A\#G^*$ -module. Then there is an exact sequence $0 \rightarrow X \rightarrow P_0 \rightarrow P_1 \rightarrow \dots$ such that $P_i \in \text{proj } A\#G^*$. Since $A\#G^*$ is a left and right free A -module, we get that $P_i \in \text{proj } A$, and then X is a Cohen-Macaulay A -module when viewing X as an A -module. We have $X \cong \bigoplus_{j=1}^t B_j^{\oplus n_j}$ and so $X \otimes_A (A\#G^*) \cong \bigoplus_{j=1}^t (B_j \otimes_A (A\#G^*))^{\oplus n_j}$. Since X is an $A\#G^*$ -summand of $X \otimes_A (A\#G^*)$, we have that X is an $A\#G^*$ -summand of $B_i \otimes_A (A\#G^*)$ for some i . Therefore, the non-isomorphic indecomposable $A\#G^*$ -summands of all the $B_i \otimes_A (A\#G^*)$ give a complete set of non-isomorphic indecomposable Cohen-Macaulay $A\#G^*$ -modules. Since this set is obviously finite, $A\#G^*$ is CM-finite.

If $A\#G^*$ is CM-finite, by the above proof, we know that $(A\#G^*)\#G$ is CM-finite too. By Blattner-Montgomery Duality Theorem, $(A\#G^*)\#G \cong M_n(A)$ which is Morita equivalent to A , where $n = |G|$. Thus A is CM-finite.

Before going on, we define a category, which is called *generic category*, whose objects are all $K[T]$ - A -bimodules which are finitely generated free as left $K[T]$ -modules and whose morphisms are $K[T]$ - A -morphisms. We denote this category by $\mathcal{GC}(A)$.

For any $X \in \mathcal{GC}(A)$, from [33, Lemma 4.3], we get that X is indecomposable in $\mathcal{GC}(A)$ if and only if $K[T]/(T - \lambda) \otimes_{K[T]} X$ is indecomposable as a $K[T]/(T - \lambda)$ - A -bimodule for any $\lambda \in K$.

If A is CM-tame, clearly $A\#G^*$ is not CM-finite. Let d be a positive integer and X an indecomposable Cohen-Macaulay $A\#G^*$ -module. Assume that $\dim_K X = d$. Then $X|X \otimes_A (A\#G^*)$. Viewing X as a A -module which is Cohen-Macaulay, assume $X = X_1 \oplus X_2 \cdots \oplus X_m$ as A -modules. Thus, X_j are Cohen-Macaulay A -modules and for some $i \in \{1, \dots, m\}$, $X|X_i \otimes_A (A\#G^*)$. Easily, $X_i \otimes_A (A\#G^*)$ are Cohen-Macaulay $A\#G^*$ -modules for all $i \in \{1, \dots, m\}$.

Since the dimension of X is finite, the dimension of X_i is finite too. Since A is CM-tame, there are finite number of $K[T]$ - A -bimodules M_j ($j = 1, \dots, n$) which are free as left $K[T]$ -modules such that all but finite number of indecomposable Cohen-Macaulay A -modules of dimension not bigger than d are isomorphic to $K[T]/(T - \lambda) \otimes_{K[T]} M_j$ for some j and some $\lambda \in K$. In order to prove that $A\#G^*$ is a CM-tame algebra, there is no harm in assuming that $X_i \cong K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j}$ for some $i_j \in \{1, \dots, n\}$ and some $\lambda \in K$. Clearly, $M_{i_j} \otimes_A (A\#G^*) \in \mathcal{GC}(A\#G^*)$.

We decompose $M_{i_j} \otimes_A (A\#G^*)$ into direct sum of indecomposable objects in $\mathcal{GC}(A\#G^*)$, $M_{i_j} \otimes_A (A\#G^*) = \bigoplus_{k \in I_j} M_{i_j}^k$ for a finite index set I_j . So $K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j}^k$ is indecomposable as an $K[T]/(T - \lambda)$ -($A\#G^*$)-bimodules too. This is equivalent to saying that $K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j}^k$ is indecomposable as an $A\#G^*$ -module since $K[T]/(T - \lambda) \cong K$. By discussions above, $X|X_i \otimes_A (A\#G^*) = K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j} \otimes_A (A\#G^*) = \bigoplus_{k \in I_j} K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j}^k$. This implies $X \cong K[T]/(T - \lambda) \otimes_{K[T]} M_{i_j}^k$ for some $i_j \in \{1, \dots, n\}$, $k \in I_j$ and $\lambda \in K$. Since the set $\{M_{i_j}^k\}_{i_j \in \{1, \dots, n\}, k \in I_j}$ is finite, $A\#G^*$ is CM-tame too.

The converse is similar to the case of CM-finiteness. We omit it here.

For the last statement, let $F_1 : \text{mod}^G A \rightarrow \text{mod} A$ be the forgetful functor. Then F_1 induces an exact functor from $\text{mod} A \# G \rightarrow A$ which is also denoted by F_1 . Since A is G -graded CM-wild, $A \#$ is CM-wild, i.e., there is a finitely generated $K\langle X, Y \rangle$ -($A \# G$)-bimodule B which is free as a left $K\langle X, Y \rangle$ -module such that the functor $- \otimes_{K\langle X, Y \rangle} B$ from $\text{mod} K\langle X, Y \rangle$, the category of finitely generated $K\langle X, Y \rangle$ -modules, to $\text{CM}(A \# G)$ preserves indecomposability and reflects isomorphisms. Combining it with F_1 , i.e., viewing B as $K\langle X, Y \rangle$ - A -bimodule, we obtain a functor $- \otimes_{K\langle X, Y \rangle} B$ from $\text{mod} K\langle X, Y \rangle$ to $\text{CM} A$, which is also preserves indecomposability, and for any $M \in \text{mod} K\langle X, Y \rangle$, there are only finitely many (in fact $|G|$) N in $K\langle X, Y \rangle$ such that $M \otimes_{K\langle X, Y \rangle} B \cong N \otimes_{K\langle X, Y \rangle} B$. Thus A is CM-semi-wild. \square

For a triple $(p, q, r) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\}$, we denote by $C(p, q, r)$ the *canonical tubular algebra* $K\Delta(p, q, r)/I(p, q, r)$ of type (p, q, r) given by the quiver $\Delta(p, q, r)$ of the form as Figure 1 shows, and the ideal $I(p, q, r)$ is generated by the element $\alpha_1\alpha_2 \cdots \alpha_p + \beta_1\beta_2 \cdots \beta_q + \gamma_1\gamma_2 \cdots \gamma_r$.

In fact, there is another canonical tubular algebra of type $(2, 2, 2, 2)$, which is not used in this paper, so we do not recall it here.

Let C be a canonical tubular algebra. From [46], we know that the Auslander-Reiten quiver Γ_C has a trisection $(\mathcal{P}^C, \mathcal{T}^C, \mathcal{Q}^C)$, where \mathcal{P}^C is a family of components containing all indecomposable projective C -modules, \mathcal{Q}^C is a family of components containing all indecomposable injective C -modules, and \mathcal{T}^C is a family $T_\lambda^C, \lambda \in \mathbb{P}_1(K)$, of stable tubes, containing all simple C -modules which are neither projective nor injective, and separating \mathcal{P}^C from \mathcal{Q}^C .

Then a *tubular algebra* (of type $\mathbb{D}_4^{(1,1)}$, $\mathbb{E}_6^{(1,1)}$, $\mathbb{E}_7^{(1,1)}$ or $\mathbb{E}_8^{(1,1)}$ respectively) is defined to be a tilted algebra $B = \text{End}_C(T)$ of a canonical tubular algebra C (of type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$ respectively), and with T a tilting C -module from the additive category $\text{add}(\mathcal{P}^C \cup \mathcal{T}^C)$. Every tubular algebra B has global dimension two and is tame of polynomial growth. We also note that the opposite algebra of a tubular algebra is also a tubular algebra.

Two tubular algebras are derived equivalent if and only if they have the same tubular type [25]. So we may speak of the tubular type of a *derived tubular algebra*, that is an algebra which is derived equivalent to a tubular algebra. Meltzer showed that each derived tubular algebra can be transformed by a finite sequence of Auslander-Platzek-Reiten (APR for short) tilts, also called sink-reflections and APR cotilts (also called source-reflections) to a canonical algebra [40]. Obviously, the opposite algebra of a derived tubular algebra is also a derived tubular algebra.

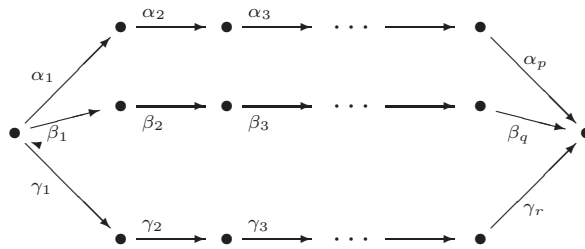


Figure 1. $\Delta(p, q, r)$

- Lemma 4.4.** (a) If Q° is of type \mathbb{D}_4 , then $T_2(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_6^{(1,1)}$;
 (b) If Q° is of type \mathbb{A}_3 , then $T_3(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_7^{(1,1)}$;
 (c) If Q° is of type \mathbb{A}_5 , then $T_2(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_8^{(1,1)}$;
 (d) If Q° is of type \mathbb{A}_2 , then $T_5(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_8^{(1,1)}$.

Proof. Since for any two derived equivalent algebras U and V , from [45, Theorem 2.1], we get that the algebras $T_m(U)$ and $T_m(V)$ are also derived equivalent for each m . So in each case, we can fix the linearly orientation for each quiver.

(a) The bound quiver of $T_2(\Lambda^\circ)$ is as the following Figure 2 shows. Then we do sink-reflection at the vertex 7, source-reflection at the vertex 4, source-reflection at the vertex 1, and the source-reflection at the vertex 3 successively, then we get a tubular algebra $\mathbb{T}_{(3,3,3)}$ [35] as Figure 3 shows.

(b) The bound quiver of $T_3(\Lambda^\circ)$ is as the following Figure 4 shows. Then we do sink-reflection at the vertex 9, source-reflection at the vertex 1, sink-reflections at the vertices 6, 8, 2, 4, and source-reflections at the vertices 3, 7 successively, then we get a tubular algebra $\mathbb{T}_{(4,4,2)}$ [35] as Figure 5 shows.

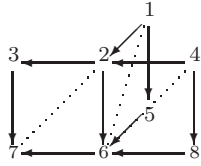


Figure 2. $T_2(\Lambda^\circ)$ with Λ° of type \mathbb{D}_4

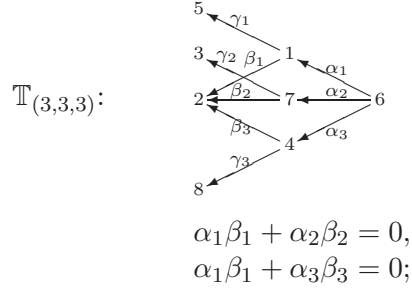


Figure 3. $\mathbb{T}_{3,3,3}$

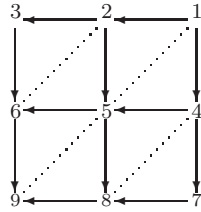


Figure 4. $T_3(\Lambda^\circ)$ with Λ° of type \mathbb{A}_3

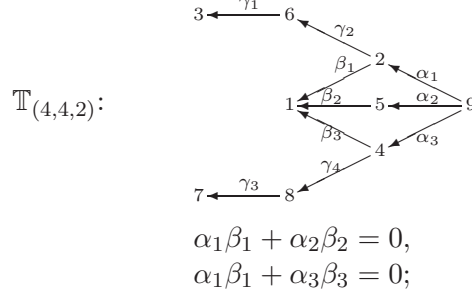


Figure 5. $\mathbb{T}_{4,4,2}$

(c) The bound quiver of $T_2(\Lambda^\circ)$ is as the following left figure shows. Then we do sink-reflection at the vertex 6, source-reflection at the vertex 5, sink-reflections at the vertices 7, 8, 1, 6, 1, source-reflection at the vertex 4, sink-reflection at the vertex 9, source-reflections at the vertices 0, 5, 0, sink-reflections at the vertices 7, 0, 5, 4, 8, source-reflections at the vertices 3, 8, 4, 9, sink-reflections at the vertices 2, 6, 2, 9, 7, 4, 8, 3, 1 successively, then we get a tubular algebra $\mathbb{T}_{(6,3,2)}$ [35] as the right figure shows.

(d) follows from case (c) directly .

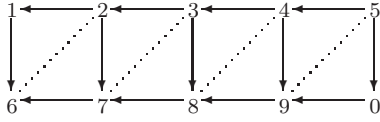
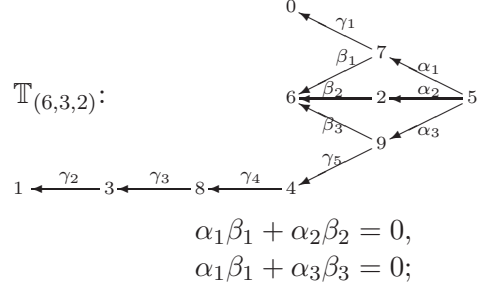
□

In particular, from [31], we know that all the algebras appearing in the Lemma 4.4 are representation tame.

Lemma 4.5. *For any $k > 0$, if $T_{k-1}(\Lambda^\circ)$ is derived equivalent to a hereditary category, then $\mathcal{R}_{T_{k-1}(\Lambda^\circ)} \simeq D^b(\text{mod } T_{k-1}(\Lambda^\circ))/\Sigma^2 \simeq D_{sg}(\text{mod }^{\mathbb{Z}/k\mathbb{Z}} H^k)$.*

Proof. Since $T_{k-1}(\Lambda^\circ)$ is derived equivalent to a hereditary algebra, $D^b(\text{mod } T_{k-1}(\Lambda^\circ))/\Sigma^2$ is triangle equivalent to $\mathcal{R}_{T_{k-1}(\Lambda^\circ)}$. The result follows from Theorem 3.10 immediately. □

For the case that Q° is of type \mathbb{A}_2 , it is proved in [47] that the forgetful functor $\text{CM}^{\mathbb{Z}} H^k \rightarrow \text{CM} H^k$ is dense if $k \leq 6$, that is, all the Cohen-Macaulay H^k -modules are gradable for $k \leq 6$.

Figure 6. $T_3(\Lambda^\circ)$ with Λ° of type \mathbb{A}_5 Figure 7. $\mathbb{T}_{6,3,2}$

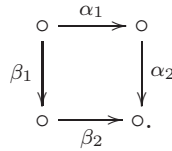
Corollary 4.6 ([47]). *If Q° is of type \mathbb{A}_2 , then H^k is CM-finite if and only if $k < 6$. In this case, the number $s(k)$ of indecomposable objects in $\text{CM}(H^k)$ is given by the formula*

$$s(k) = 2 + 2(k-1) \frac{6}{6-k}.$$

Proof. For $k = 1$, then $H^1 = \Lambda^\circ = KQ^\circ$ is a hereditary algebra of type \mathbb{A}_2 , which implies H^1 is CM-free, and $s(1) = 2$.

For $k = 2$, then $T_2(\Lambda^\circ) = K(\circ \rightarrow \circ)$ which is a hereditary algebra of type \mathbb{A}_2 . So H^2 is CM-finite since $T_2(\Lambda^\circ)$ is representation finite and $\underline{\text{CM}}(H^2) \simeq D^b(\text{mod } T_2(\Lambda^\circ))/\Sigma$. In particular, it is easy to see that $s(2) = 2 + 3 = 5$ since the number of indecomposable objects in $D^b(\text{mod } T_2(\Lambda^\circ))/\Sigma$ is 3.

For $k = 3$, then $T_2(\Lambda^\circ) = KQ^2/J^2$ where Q is the quiver as the following diagrams shows, and $J^2 = \langle \alpha_1\alpha_2 - \beta_1\beta_2 \rangle$.



Then $T_2(\Lambda^\circ)$ is a tilted algebra of type \mathbb{D}_4 . So $\underline{\text{CM}}^{\mathbb{Z}/3\mathbb{Z}} H^3 \simeq \mathcal{R}_{T_2(\Lambda^\circ)}$ which is triangle equivalent to the root category of type \mathbb{D}_4 . Then the $\mathbb{Z}/3\mathbb{Z}$ -graded algebra H^3 is graded CM-finite. From Theorem 4.3, we get that H^3 is CM-finite. So 3 times of $s(3) - 2$ is equal to the number of indecomposable objects in $(\underline{\text{CM}}^{\mathbb{Z}} H^3)/(3) \simeq D^b(\text{mod } T_2(\Lambda^\circ))/\Sigma^2$, which is 24, i.e. 2 times of the number of indecomposable modules for type \mathbb{D}_4 . So $s(3) = 2 + 8$.

For $k = 4, 5$, it is similar to the case $k = 3$, only need to note that $T_3(\Lambda^\circ)$ and $T_4(\Lambda^\circ)$ are tilted algebras of type \mathbb{E}_6 and \mathbb{E}_8 respectively. \square

Corollary 4.7 ([47]). *If Q° is of type \mathbb{A}_2 , then H^6 is CM-tame and H^k is CM-semi-wild for $k > 6$.*

Proof. Since Q° is of type \mathbb{A}_2 , Lemma 4.4 yields that $T_5(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_8^{(1,1)}$. Then the root category $\mathcal{R}_{T_5(\Lambda^\circ)}$ is triangle equivalent to $D^b(\text{mod } T_5(\Lambda^\circ))/\Sigma^2$. For H^6 , we get that $\underline{\text{CM}}^{\mathbb{Z}/6\mathbb{Z}}(H^6) \simeq \mathcal{R}_{T_5(\Lambda^\circ)} \simeq D^b(\text{mod } T_5(\Lambda^\circ))/\Sigma^2$. Since $T_5(\Lambda^\circ)$ is a derived tubular algebra of type $\mathbb{E}_8^{(1,1)}$, from the structure of its derived category described in [25], we get that H^6 is a $\mathbb{Z}/6\mathbb{Z}$ -graded CM-tame algebra. Then Theorem 4.3 shows that H^6 is CM-tame.

For H^k with $k > 6$, from [32, Proposition 1], we know that $T_{k-1}(\Lambda^\circ)$ is a wild algebra. Since $\text{Ind}(\text{mod } T_{k-1}(\Lambda^\circ))$ is a subset of $\text{Ind}(D^b(\text{mod } T_{k-1}(\Lambda^\circ))/\Sigma^2)$ and $\underline{\text{CM}}^{\mathbb{Z}/k\mathbb{Z}}(H^k) \simeq \mathcal{R}_{T_{k-1}(\Lambda^\circ)}$, it is easy to see that H^k is a $\mathbb{Z}/k\mathbb{Z}$ -graded CM-wild algebra. From Theorem 4.3, we get that H^k is CM-semi-wild for $k > 6$. \square

Proposition 4.8. *For any integer $k > 0$, H^k is CM-finite if and only if it belongs to one of the following cases:*

- (a) $k = 1$;
- (b) $k = 2$, Q° is of Dynkin type \mathbb{A}, \mathbb{D} , or \mathbb{E} ;
- (c) $k = 3$, Q° is of type $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ or \mathbb{A}_4 ;
- (d) $k = 4$ or 5 , Q° is of type $\mathbb{A}_1, \mathbb{A}_2$;
- (e) $k \geq 6$, Q° is of type \mathbb{A}_1 .

Proof. It is easy to see that the algebras $T_{k-1}(\Lambda^\circ)$ appearing in (a-e) are either hereditary algebras or tilted algebras of Dynkin type. From $\underline{\text{CM}}^{\mathbb{Z}/k\mathbb{Z}} H^k \simeq D^b(\text{mod } T_{k-1}(\Lambda^\circ))/\Sigma^2$, we get that H^k is $\mathbb{Z}/k\mathbb{Z}$ -graded CM-finite and then it is CM-finite in each (a-e).

Conversely, since H^k is CM-finite, obviously, $T_{k-1}(\Lambda^\circ)$ is representation finite, which implies that Λ° is also representation finite. For $k = 1$, it is obvious that H^1 is always CM-finite. For $k = 2$, $\text{CM } H^k \simeq D^b(\text{mod } KQ^\circ)/\Sigma$, so H^k is CM-finite only if Q° is of Dynkin type \mathbb{A}, \mathbb{D} , or \mathbb{E} . For $k = 3$, if Q° is of type \mathbb{A}_n for $n \geq 5$, \mathbb{D} or \mathbb{E} , then $T_2(\Lambda^\circ)$ is a representation infinite algebra by [31, Section 5], so H^k is not CM-finite in this case. For $k = 4, 5$, if Q° contains a subquiver of type \mathbb{A}_3 , from Lemma 4.4 (b), it is easy to see that $T_{k-1}(\Lambda^\circ)$ is representation infinite. For $k \geq 6$, if Q° contains a subquiver of type \mathbb{A}_2 , then $T_{k-1}(\Lambda^\circ)$ is representation infinite by Lemma 4.4 (d). \square

Proposition 4.9. *For any integer $k > 0$, H^k is CM-tame if and only if it belongs to one of the following cases:*

- (a) $k = 2$, Q° is of Euclidean type $\tilde{\mathbb{A}}, \tilde{\mathbb{D}}, \tilde{\mathbb{E}}$;
- (b) $k = 3$, Q° is of type $\mathbb{D}_4, \mathbb{A}_5$;
- (c) $k = 4$, Q° is of type \mathbb{A}_3 ;
- (d) $k = 5$, Q° is of type \mathbb{A}_2 ;

Proof. First, from Lemma 4.4, it is easy to see that all algebras H^k appearing in (a-d) are CM-tame. Conversely, for $k = 1$, H^k has no non-projective Cohen-Macaulay modules, so it is not CM-tame for any Q° . For $k = 2$, $\text{CM } H^2 \simeq D^b(\text{mod } KQ^\circ)/\Sigma$, so H^2 is CM-tame if and only if Q° is of Euclidean type $\tilde{\mathbb{A}}, \tilde{\mathbb{D}}, \tilde{\mathbb{E}}$.

For $k \geq 3$, since H^k is CM-tame, we get that $T_{k-1}(\Lambda^\circ)$ is representation tame, which implies that Λ° is of Dynkin type, see [30, 50, 32]. For $k = 3$, if Q° contains a subquiver of type \mathbb{A}_6 , then $T_2(\Lambda^\circ)$ is wild, which implies that H^3 is $\mathbb{Z}/3\mathbb{Z}$ -graded CM-wild and then not $\mathbb{Z}/3\mathbb{Z}$ -graded CM-tame; if Q° contains a subquiver of type \mathbb{D}_5 , then [31, Proposition 1] shows that $T_2(\Lambda^\circ)$ is representation wild, so H^3 is $\mathbb{Z}/3\mathbb{Z}$ -graded CM-wild and then not $\mathbb{Z}/3\mathbb{Z}$ -graded CM-tame. From [32, Theorem 6.2], we get that for $k \geq 4$, $T_k(\Lambda^\circ)$ is representation tame if and only if $k = 4$ and Q° is of type \mathbb{A}_3 , or $k = 5$ and Q° is of type \mathbb{A}_2 . In both cases, they are derived tubular algebra by Lemma 4.4, which implies our desired result immediately. \square

Corollary 4.10. *For any integer $k > 0$, H^k is in one and only one of the cases: CM-finite, CM-tame and CM-semi-wild.*

Proof. From Propositions 4.8 and 4.9, we have classified H^k up to CM-finiteness and CM-tameness. From their proof, we get that if H^k is not CM-finite or CM-tame, then $T_{k-1}(\Lambda^\circ)$ is

representation wild, which implies that H^k is $\mathbb{Z}/k\mathbb{Z}$ -graded CM-wild, and then it is CM-semi-wild. \square

5. EXAMPLES

In this section, we describe the Auslander-Reiten quiver of $\text{CM } H^k$ for some H^k of Cohen-Macaulay finite type. For the algebras H^k appearing in Proposition 4.8, the cases (a) and (e) are trivial. For case (b), the Auslander-Reiten quiver of $\text{CM } H^2$ is described in [48]. For case (c) $k = 3, 4, 5$, when Q° is of type \mathbb{A}_2 , its Auslander-Reiten quiver is described in [47]. So we only need to describe the Auslander-Reiten quiver for $k = 3$, and Q° of type \mathbb{A}_3 and \mathbb{A}_4 . In this section, we describe the Auslander-Reiten quivers for them with Q° linearly oriented.

5.1. Let $H^3 = KQ/I$ with the bound quiver (Q, I) as Figure 8 shows, where I is generated by the relation $\varepsilon_1^3 = \varepsilon_2^3 = \varepsilon_3^3 = 0$, $\beta\varepsilon_2 = \varepsilon_3\beta$ and $\alpha\varepsilon_1 = \varepsilon_2\alpha$.

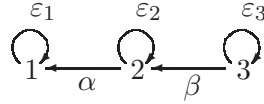


Figure 8. H^3 with Λ° hereditary of type \mathbb{A}_3

The Auslander-Reiten quiver of H^3 is displayed in Figure 9. As vertices we have the graded dimension vectors (arising from the obvious \mathbb{Z} -covering of H^3) of the indecomposable H^3 -modules. The modules on the leftmost column of the figure has to be identified with the corresponding module on the rightmost one. The projective H^3 -modules are marked with a solid frame.

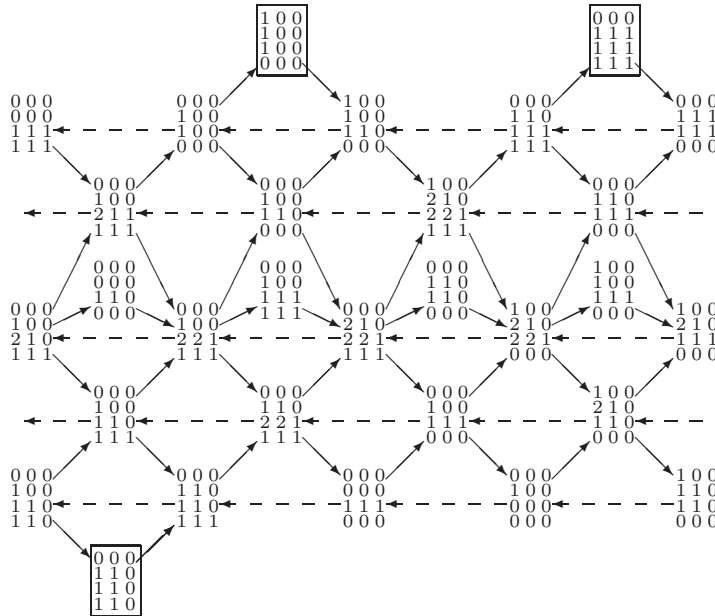


Figure 9. Auslander-Reiten quiver for H^3 with Λ° hereditary of type \mathbb{A}_3

5.2. Let Q be given by the quiver as Figure 10 shows.

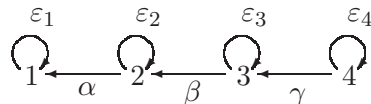
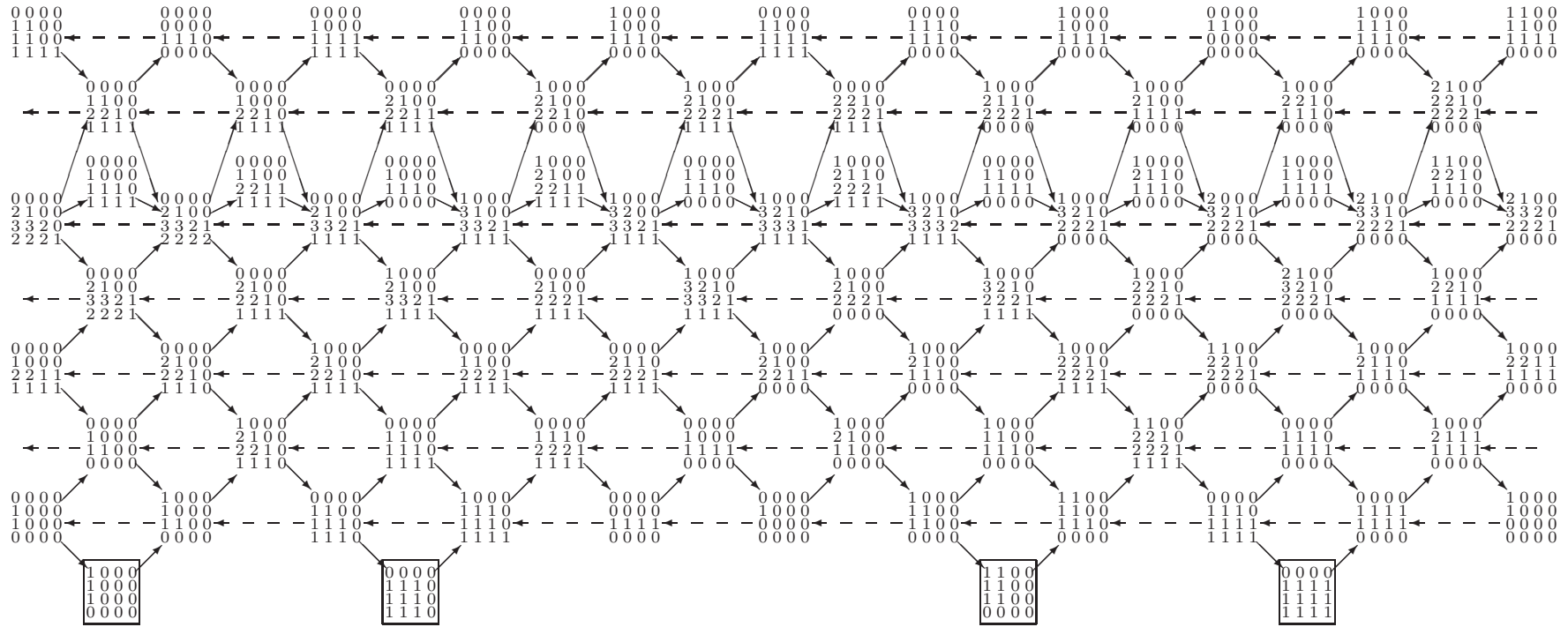


Figure 10. H^3 with Λ° hereditary of type \mathbb{A}_4

Figure 11. Auslander-Reiten quiver of H^3 with Λ° hereditary of type \mathbb{A}_4

Let $H^3 = KQ/I$ with I generated by the relation $\varepsilon_1^3 = \varepsilon_2^3 = \varepsilon_3^3 = \varepsilon_4^3 = 0$, $\alpha\varepsilon_1 = \varepsilon_2\alpha$, $\beta\varepsilon_2 = \varepsilon_3\beta$ and $\gamma\varepsilon_3 = \varepsilon_4\gamma$. The Auslander-Reiten quiver of H^3 is displayed in Figure 11. The modules on the leftmost column of the figure has to be identified with the corresponding module on the rightmost one.

5.3. Finally, we give an example with Λ° not hereditary. Let Q be given by the following quiver.

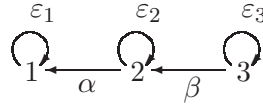


Figure 12. The quiver of H_1^2

Let $H_1^2 = KQ/I_1$ with I_1 generated by the relation $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 0$, $\beta\alpha = 0$, $\beta\varepsilon_2 = \varepsilon_3\beta$ and $\alpha\varepsilon_1 = \varepsilon_2\alpha$. The Auslander-Reiten quiver of H_1^2 is displayed in Figure 13. The modules on the leftmost column of the figure has to be identified with the corresponding module on the rightmost one.

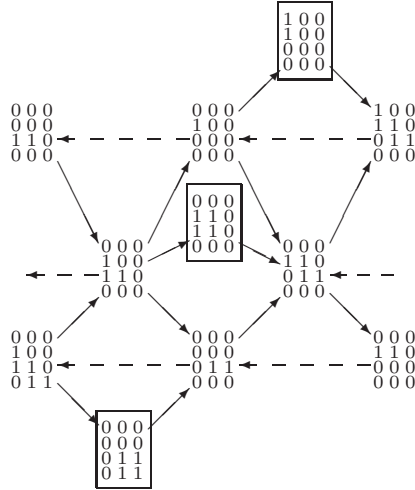


Figure 13. Auslander-Reiten quiver for H_1^2

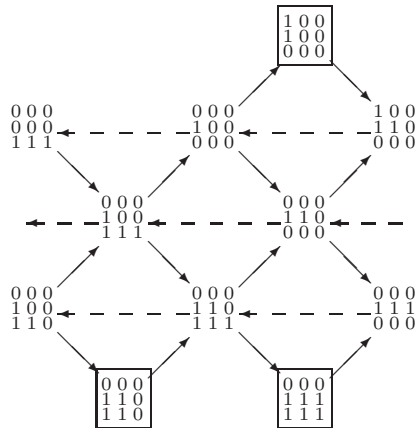


Figure 14. Auslander-Reiten quiver for H_2^2

As a comparison, let $H_2^2 = KQ/I_2$ with I_2 generated by the relation $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 0$, $\beta\varepsilon_2 = \varepsilon_3\beta$ and $\alpha\varepsilon_1 = \varepsilon_2\alpha$. The Auslander-Reiten quiver of H_2^2 is displayed in Figure 14. The modules on the leftmost column of the figure has to be identified with the corresponding module on the rightmost one.

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DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610064, P.R.CHINA

E-mail address: luming@scu.edu.cn